JOURNAL OF FUNCTIONAL ANALYSIS 45, 139–168 (1982)

$C(\alpha)$ Preserving Operators on Separable Banach Spaces*

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Communicated by the Editors

Received January 30, 1980; revised April 11, 1981

Let T be a bounded linear operator from a separable Banach space X to a Banach space Y. A necessary and sufficient condition on T for the existence of a subspace Z of X such that Z is isomorphic to $C(\alpha)$ and the restriction of T to Z is an isomorphism is given. The special case where X is the disc algebra is then considered and results similar to those previously obtained by the author for $C(K)$ spaces are obtained for the disc algebra. Finally some additional results of the same type are proved for subspaces of $C(K)$ with small annihilator.

0. INTRODUCTION

Our goal in this paper is to prove results on bounded linear operators with domain a separable Banach space analogous to those obtained in [1] for operators with domain a $C(K)$ space. The approach is to replace the notion of $\hat{\alpha}$-family of sets by the notion of $\hat{\alpha}$-family of functions. We define $\hat{\alpha}$-family of functions in Section 1 and describe the connection between $\hat{\alpha}$-families of functions and operators on $C_0(\omega^\alpha)$. Once this is established we employ the result of [1] to prove our characterization of $C_0(\omega^\alpha)$ preserving operators.

In Section 2 we consider the special case of the disc algebra $\mathscr{A}$ and show that there are enough functions in $\mathscr{A}$ to characterize $C_0(\omega^\alpha)$ preserving operators with domain $\mathscr{A}$ by $\hat{\alpha}$-families of sets. In the third section we extend the techniques to subspaces of $C(K)$ with separable annihilator and obtain similar results provided the annihilator is sufficiently small.

We will use standard Banach space notation as may be found in the book of Lindenstrauss and Tzafriri [15]. $C(K)$ will denote the space of complex valued functions on a compact Hausdorff space K. If α is an ordinal $C(\alpha)$ (resp. $C_0(\alpha)$), will denote the space of continuous functions on the ordinals

* Supported in part by NSF MCS-7910272.

0022-1236/82/020139–30\$02.00/0
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less than or equal to α with the order topology (resp. and vanishing at α). If $\mathscr{A}$ is a topological space, $\mathscr{A}^{(\alpha)}$ will denote the αth derived set of $\mathscr{A}$. We will use interval notation with sets of ordinals with the usual conventions, e.g., $\{\gamma: \alpha < \gamma < \beta\} = (\alpha, \beta)$. If Λ is a set of ordinals, $\mathscr{A}$ is a subset of $C(K)^*$, and we write $\mathscr{A} = \{\mu_\alpha : \alpha \in \Lambda\}$ then we mean that the map $\alpha \to \mu_\alpha$ is a homeomorphism where $\mathscr{A}$ has the w^* topology and Λ has the topology inherited from $[1, \sup \Lambda]$ with the order topology.

Much of the work here depends on the results of [1]. Let us note that although the arguments given there applied to the real case, only minor modifications are necessary to obtain the same results for complex valued functions.

We will now give the definition and basic properties of $\hat{\gamma}$-families of sets.

DEFINITION. Let $\gamma < \omega_1$ and let K be a compact Hausdorff space. A family $\mathscr{F}$ of nonempty open subsets of K is a *$\hat{\gamma}$-family* if for each α with $0 \leqslant \alpha < \gamma$ there is a subfamily $\mathscr{F}_\alpha$ of $\mathscr{F}$ such that $\mathscr{F}$ has the following properties:

(0) If $G_i \in \mathscr{F}_{\alpha_i}$, $i = 1, 2$, $G_1 \cap G_2 = \varnothing$; $G_1 \subset G_2$ and $\alpha_1 < \alpha_2$; or $G_2 \subset G_1$ and $\alpha_2 < \alpha_1$.

(1) $\mathscr{F} = \bigcup \{\mathscr{F}_\alpha : \alpha < \gamma\}$.

(2) If $\alpha < \gamma$, $\mathscr{F}_\alpha$ is an infinite family of disjoint open sets.

(3) If $G \in \mathscr{F}_\beta$ and $0 \leqslant \alpha < \beta < \gamma$, the set $\{H: H \subset G \text{ and } H \in \mathscr{F}_\alpha\}$ is infinite.

(4) If $G \in \mathscr{F}_{\beta+1}$ then there is a sequence $\{G_n : n \in N\}$ of disjoint sets in $\mathscr{F}_\beta$ such that $\overline{\bigcup G_n} \subset G$.

(5) If $G \in \mathscr{F}_\beta$ and β is a limit ordinal, then there exists a sequence $\beta_n \uparrow \beta$ and a sequence $\{G_n : n \in N\}$ of disjoint sets such that $G_n \in \mathscr{F}_{\beta_n}$ for each n and $\overline{\bigcup G_n} \subset G$.

If there is also a set of measures $\mathscr{M}$ on K associated with $\mathscr{F}$, i.e., for each $G \in \mathscr{F}$ there is a measure $\mu_G \in \mathscr{M}$, which satisfies:

(4′) If $G \in \mathscr{F}_{\beta+1}$, then there is a sequence $\{G_n : n \in N\}$ of disjoint sets with $G_n \in \mathscr{F}_\beta$ for all n, $\overline{\bigcup G_n} \subset G$, and $\mu_{G_n} \to^{w^*} \mu_G$.

(5′) If $G \in \mathscr{F}_\beta$, where β is a limit ordinal, then there exists a sequence $\{G_n : n \in N\}$ of disjoint subsets of G and $\beta_n \uparrow \beta$ such that $G_n \in \mathscr{F}_{\beta_n}$, $\mu_{G_n} \to^{w^*} \mu_G$, and $\overline{\bigcup G_n} \subset G$.

(6) For each $G \in \mathscr{F}$, $|\mu_G|\,(G) \geqslant \varepsilon$, then the $\hat{\gamma}$-family is said to have *ε-measures.*

(Note that (0) is slightly stronger than in [1]).

We will also use a slight modification of this notion of $\hat{\gamma}$-family. A γ-family $\mathscr{F}$ has the same properties as those of a $\hat{\gamma}$-family except that there is an open set G_γ such that $G_\gamma \supset \bigcup \mathscr{F}$ and if there are measures associated with $\mathscr{F}$ then there exists a sequence of disjoint sets $\{G_n : n \in N\} \subset \mathscr{F}$ and ordinals $\gamma_n \uparrow \gamma$ such that $G_n \in \mathscr{F}_{\gamma_n}$ (or if γ is a successor, $G_n \in \mathscr{F}_{\gamma-1}$) and $\mu_{G_n} \to^{w^*} \mu_{G_\gamma}$.

In our previous work we made use of an index which was related to the notion of $\hat{\gamma}$-family by Lemma 1.0 of [1]. Here we will use only the $\hat{\gamma}$-families, however, the reader may find it necessary to translate the results of [1] that were stated in terms of the index by using that lemma. We will also use the fact that every $\hat{\gamma}$-family $\mathscr{F}$ with ε-measures $\mathscr{M}$ contains a $\hat{\gamma}$-family $\mathscr{G} = \{G_\alpha : \alpha < \omega^\gamma\}$ with ε-measures $\{\mu_\alpha : \alpha < \omega^\gamma\}$ in $\mathscr{M}$ $(\mu_\alpha = \mu_{G_\alpha})$ such that for each $\alpha < \omega^\gamma$,

$$
\begin{aligned}
&\overline{\bigcup \{G_\tau : G_\tau \subsetneq G_\alpha\}} \subset G_\alpha\,,\\
&\text{there is a neighborhood } \mathscr{N}_\alpha \text{ of } \alpha \text{ in } [1, \omega^\gamma) \text{ such that} \qquad\qquad (0.1)\\
&G_\alpha \supset \overline{\{G_\tau : \tau \in \mathscr{N}_\alpha - \{\alpha\}\}}, \text{ and } \{\tau \colon G_\tau \supseteq G_\alpha\} \text{ is finite,}
\end{aligned}
$$

to assume that any $\hat{\gamma}$-family we use has been refined to have these properties. We will also use the following (Lemma 1.1 of [1]):

LEMMA 0.1. *Let $\mathscr{F} = \{G_\alpha \colon \alpha \leqslant \omega^\gamma\}$ be a γ-family with ε-measures $\{\mu_\alpha : \alpha \leqslant \omega^\gamma\}$ satisfying* (0.1). *If A is a closed subset of $[1, \omega^\gamma]$ and A is homeomorphic to $[1, \omega^\beta]$, then $\mathscr{G} = \{G_\alpha \colon \alpha \in A\}$ contains a β-family with ε-measures in $\{\mu_\alpha \colon \alpha \in \mathscr{A}\}$.*

1. THE GENERAL CASE

In order to describe $C(\alpha)$ preserving operators on a separable Banach space X we first need to study the image of $C(\alpha)$ under an operator T: $C(\alpha) \to X$. Our first lemma will give us a formula for computing the norm of such an operator based on the images under T of a certain family of characteristic functions in $C(\alpha)$. Recall that a topological space is *dispersed* if it contains no nonempty perfect subset and that if K is a dispersed compact Hausdorff space, then $C(K)^*$ is isometric to $l_1(K)$ (and conversely) [20].

We wish to thank J. Wolfe for several useful suggestions which led to the present versions of Lemmas 1.1 and 1.2.

LEMMA 1.1. *Let K be a dispersed compact Hausdorff space and let Ω be a set. Let G be a family of clopen subsets of K such that*

(i) *if* $G_1, G_2 \in \mathscr{G}$, *then* $G_1 \cap G_2 = \varnothing$, $G_1 \subset G_2$ *or* $G_2 \subset G_1$,

(ii) $\{1_G\colon G \in \mathscr{G}\}$ *separates the points of* K,

(iii) *for each* $k \in K$, $\{G\colon G \in \mathscr{G},\ k \in G\}$ *is well ordered by inclusion.*

Let $Y = \overline{\operatorname{span}}\{1_G\colon G \in \mathscr{G}\}$ *and let* T *be a linear operator from* Y *into* $l_\infty(\Omega)$. *Then* $\|T\| = \|\sum_{G\in\mathscr{G}} |T1_G - \lim_{\mathscr{H}\in\mathscr{H}(G)} \sum_{H\in\mathscr{H}} T1_H| \|_\infty$, *where the convergence of the sums is pointwise and the limit is over the directed set* $\mathscr{H}(G) = \{\mathscr{H} \subset \mathscr{G}\colon \mathscr{H}$ *is a maximal family of disjoint proper subsets of* $G\}$ *ordered by* $\mathscr{H} \leqslant \mathscr{H}'$ *if for each* $H \in \mathscr{H}$ *there is an* $H' \in \mathscr{H}'$ *such that* $H \subset H'$.

Proof. Assumptions (i) and (ii) and the Stone Weierstrauss Theorem show that $Y = C(K)$ or that there is a $k_0 \in K$ such that $Y = \{f \in C(K)\colon f(k_0) = 0\}$. In either case Y^* can be identified with $l_1(\Gamma)$, where $\Gamma = K$ or $K - \{k_0\}$, respectively. For each $\gamma \in \Gamma$ by (iii) there is a minimal set $G \in \mathscr{G}$ such that $\gamma \in G$ and it follows from (ii) that $\{\gamma\} = G - \bigcup \{H\colon H \not\subseteq G, H \in \mathscr{G}\}$. Hence if δ_ω denotes the evaluation at a point $\omega \in \Omega$,

$$\begin{aligned}\|T^*\delta_\omega\| &= \sum_{G\in\mathscr{G}} \left| T^*\delta_\omega \left(G - \bigcup \{H\colon H \not\subseteq G, H \in \mathscr{G}\}\right)\right| \\ &= \sum_{G\in\mathscr{G}} \left| \delta_\omega(T1_G) - T^*\delta_\omega \left[\bigcup \left(\bigcup_{\mathscr{H}\in\mathscr{H}(G)} \mathscr{H}\right)\right]\right| \\ &= \sum_{G\in\mathscr{G}} \left| \delta_\omega(T1_G) - \lim_{\mathscr{H}\in\mathscr{H}(G)} T^*\delta_\omega \left(\bigcup \mathscr{H}\right)\right| \\ &= \sum_{G\in\mathscr{G}} \left| \delta_\omega(T1_G) - \lim_{\mathscr{H}\in\mathscr{H}(G)} \sum_{H\in\mathscr{H}} \delta_\omega(T1_H)\right|\end{aligned}$$

proving the result.

LEMMA 1.2. *Let* K, Ω, $\mathscr{G}$ *and* T *be as in Lemma* 1.1. *Assume further that for each* $k \in K$, $\{G\colon k \in G\}$ *ordered by inclusion is of order type less than or equal to* ω. *Then*

$$\|T\| = \left\| \sum_{G\in\mathscr{G}} \left| T1_G - \sum_{H\Box G} T1_H \right| \right\|_\infty,$$

where the convergence of the sums is pointwise and $H \Box G$ *if and only if* $H \in \mathscr{G}$ *and* H *is a maximal proper subset of* G.

Proof. Observe that the directed set $\mathscr{H}(G)$ has a terminal element, $\{H\colon H \Box G\}$, for each $G \in \mathscr{G}$.

It is easy to see that a $\hat{\gamma}$-family $\mathscr{G}$ satisfying (0.1) has the property that any nested family of sets in $\mathscr{G}$ is finite. Indeed, by property (0) such a family is well ordered by inclusion and by (0.1) no set in $\mathscr{G}$ can be contained in an infinite increasing nest of sets. Thus a $\hat{\gamma}$-family of subsets of a compact Hausdorff space K will satisfy both (i) of Lemma 1.1 and the strengthened version of (iii) of Lemma 1.2. The next lemma shows that for $K = [1, \omega^\gamma]$, $\gamma < \omega_1$, (ii) can also be satisfied.

LEMMA 1.3. *For any ordinal* $1 \leqslant \gamma < \omega_1$ *there is a* $\hat{\gamma}$*-family* $\mathscr{G}$ *of subsets of* $[1, \omega^\gamma)$ *satisfying* (0.1) *such that* $C_0(\omega^\gamma) = \overline{\operatorname{span}}\{1_G : G \in \mathscr{G}\}$.

Proof. If $\gamma = 1$, let $\mathscr{G} = \{\{n\} : n < \omega\}$. Inductively assume that the lemma has been proved for all $\alpha < \gamma$. If $\gamma = \beta + 1$ for some ordinal β, let $\mathscr{G} = \bigcup \{\mathscr{G}_n : n < \omega\} \cup \{(\omega^\beta(n-1), \omega^\beta n] : n < \omega\}$, where for each n, $\mathscr{G}_n$ is a $\hat{\beta}$-family of subsets of $(\omega^\beta(n-1), \omega^\beta n]$ satisfying (0.1) such that $C_0(\omega^\beta(n-1), \omega^\beta n] = \overline{\operatorname{span}}\{1_G : G \in \mathscr{G}_n\}$. If γ is a limit ordinal let $\beta_n \uparrow \gamma$ and let $\mathscr{G} = \bigcup \{\mathscr{G}_n : n < \omega\} \cup \{(\omega^{\beta_{n-1}}, \omega^{\beta_n}] : n < \omega\}$, where $\mathscr{G}_n$ is a $\hat{\beta}_n$-family of subsets of $(\omega^{\beta_{n-1}}, \omega^{\beta_n}]$ satisfying (0.1) such that $C_0(\omega^{\beta_{n-1}}, \omega^{\beta_n}] = \overline{\operatorname{span}}\{1_G : G \in \mathscr{G}_n\}$. We leave the verification to the reader.

In the following definition we refer to a $\hat{\gamma}$-family $\mathscr{G}$ of subsets of a set Γ instead of a compact Hausdorff space K. By this we mean that in $\beta\Gamma$, the Stone-Čech compactification, $\{\bar{G} : G \in \mathscr{G}\}$ is a $\hat{\gamma}$-family.

DEFINITION. Let $\mathscr{G}$ be a $\hat{\gamma}$-family of subsets of a set Γ and let ϕ be a map from $\mathscr{G}$ into a Banach space X. The pair $(\phi, \mathscr{G})$ will be called a $\hat{\gamma}$*-family of functions in* X. (Sometimes we will refer to the range of ϕ as the $\hat{\gamma}$-family of functions.) A $\hat{\gamma}$-family of functions in X will be said to be λ*-bounded* if the linear extension $\tilde{\phi}$ of ϕ has norm less than or equal to λ, i.e., if $Z = \overline{\operatorname{span}}\{1_G : G \in \mathscr{G}\}$ in $l_\infty(\Gamma)$, $\tilde{\phi}: Z \to X$ by $\tilde{\phi}(\sum a_i 1_{G_i}) = \sum a_i \phi(G_i)$, and $\|\tilde{\phi}\| \leqslant \lambda$. Finally if there is a mapping $\psi: \mathscr{G} \to X^*$ and $\delta > 0$ such that

(a) If $G \in \mathscr{G}_{\beta+1}$, then there is a sequence $\{G_n : n \in \mathbb{N}\}$ of disjoint sets with $G_n \in \mathscr{F}_\beta$ for all n, $\overline{\bigcup G_n} \subset G$, and $\psi(G_n) \to^{w^*} \psi(G)$.

(b) If $G \in \mathscr{G}_\beta$, where β is a limit ordinal, then there exists a sequence $\{G_n : n \in \mathbb{N}\}$ of disjoints subsets of G and $\beta_n \uparrow \beta$ such that $G_n \in \mathscr{G}_{\beta_n}$, $\overline{\bigcup G_n} \subset G$, and $\psi(G_n) \to^{w^*} \psi(G)$.

(c) $\psi(G)[\phi(G)] \geqslant \delta$ for all $G \in \mathscr{G}$ we will say that the $\hat{\gamma}$-family of functions has δ-measures.

(Compare with (4′), (5′) and (6) of the definition of a $\hat{\gamma}$-family of sets with ε-measures.)

If follows from Lemmas 1.1–1.3 that if $K = [1, \omega^\alpha]$ and $\mathscr{G}$ is the $\hat{\alpha}$-family from Lemma 1.3 then $\tilde{\phi}$ is λ bounded if and only if $\sum_{G \in \mathscr{G}} |\phi(G) - \sum_{H \sqsubset G} \phi(H)|\,(x^*) \leqslant \lambda$ for all $x^* \in B_{X^*}$. Our next result shows that this formula holds for any $\hat{\alpha}$-family $\mathscr{G}$ satisfying (0.1).

LEMMA 1.4. *If $\mathscr{G}$ is an $\hat{\alpha}$-family of subsets of a set Γ satisfying* (0.1) *then there is an $\hat{\alpha}$-family $\mathscr{F}$ of subsets of $[1, \omega^\alpha)$ and a Boolean algebra isomorphism ξ from the algebra generated by $\mathscr{G}$ to the algebra generated by $\mathscr{F}$ such that $\xi(G) \in \mathscr{F}$ for all $G \in \mathscr{G}$ and $\xi(\mathscr{G}) = \mathscr{F}$. Moreover, $C_0(\omega^\alpha) = \overline{\operatorname{span}}\{1_F : F \in \mathscr{F}\}$.*

Proof. We will use induction on α. If $\alpha = 1$, $\mathscr{G}$ is a countable family of disjoint sets and thus can be put into one to one correspondence with $\{\{n\}: n < \omega\}$. Next assume that the lemma is true for all $\gamma < \alpha$ and let $\{G_n : n < \omega\}$ be a maximal family of maximal sets in $\mathscr{G}$, i.e., if $G \in \mathscr{G}$ there is an n such that $G \subset G_n$ and $G_n \cap G_{n'} = \varnothing$ for all $n \neq n'$. Observe that $\mathscr{B}_n = \{G : G \not\subseteq G_n, G \in \mathscr{G}\}$ is an $\hat{\alpha}_n$-family for $\alpha_n < \alpha$ for each n and thus by induction there are Boolean algebra isomorphisms $\xi_n : \mathscr{B}_n \to \mathscr{F}_n$, where $\mathscr{F}_n$ is an $\hat{\alpha}_n$-family of subsets of $[1, \omega^{\alpha_n})$. Clearly $\sup \alpha_n = \alpha$ and $\alpha_n < \alpha$, for all n, if α is a limit ordinal and if $\alpha = \beta + 1$, $\alpha_n = \beta$ for infinitely many n. Thus we can find a sequence $\gamma_n \uparrow \omega^\alpha$ such that $(\gamma_{n-1}, \gamma_n]$ is homeomorphic to $[1, \omega^{\alpha_n}]$ with homeomorphism $\zeta_n : [1, \omega^{\alpha_n}] \to (\gamma_{n-1}, \gamma_n]$. Clearly if we define $\xi(G_n) = (\gamma_{n-1}, \gamma_n]$ for all n, $\xi(G) = \zeta_n(\xi_n(G))$ for all $G \in \mathscr{B}_n$, $n < \omega$, we can generate the required Boolean isomorphism as an extension of ξ.

Remark 1.1. Using the above results we could write the definition of a λ-bounded $\hat{\gamma}$-family of functions in X with δ-measures without reference to the notion of $\hat{\gamma}$ family of sets since we use the notion of $\hat{\gamma}$-family of sets only as an index. We leave this to the interested reader. We do, however, wish to point out that a λ-bounded $\hat{1}$-family of functions in X is a weakly unconditionally Cauchy sequence and if there are ε-measures in B_{X^*} for this $\hat{1}$-family, it is not unconditionally converging. (See [3] or [15].)

PROPOSITION 1.5. *Let X be a separable Banach space and let B be a w^* closed convex symmetric subset of B_{X^*}. A necessary and sufficient condition for B to norm a subspace of X isomorphic to $C_0(\omega^{\omega^\alpha})$ is that there exist constants $0 < \delta$, $\lambda < \infty$ and a λ-bounded $\widehat{\omega^\alpha}$-family of functions in X with δ-measures in B.*

Proof. Let Y be a subspace of X isomorphic to $C_0(\omega^{\omega^\alpha})$ and let $T: C_0(\omega^{\omega^\alpha}) \to X$ be an isomorphism onto Y. If B norms Y, it follows from the Hahn–Banach Theorem that $T^*B \supset \rho B_{C_0(\omega^{\omega^\alpha})^*}$ for some $\rho > 0$. By Proposition 2 of [15] there is a subset A of B such that $\rho^{-1}T^*$ is a homeomorphism from A onto some closed subset S of $\{\delta_\beta : \beta < \omega^{\omega^\alpha}\} \cup \{0\}$ with $S^{(\omega^\alpha)} \neq \varnothing$. By Lemma 1.3 of [1] there is a positive linear isometry L from $C_0(S)$ into $C_0(\omega^{\omega^\alpha})$ such that $L^*s = \delta_s$ for all $s \in S$. Hence $\rho^{-1}L^*T^*$ is a homeomorphism of A onto $\{\delta_s : s \in S\} \subset B_{C(S)^*}$ and $\rho^{-1}TL$ is an isomorphism of $C_0(S)$ into $Y \subset X$. Let $\mathscr{G}$ be the $\widehat{\omega^\alpha}$-family of subsets of S satisfying (0.1) such that $\overline{\operatorname{span}}\{1_G : G \in \mathscr{G}\} = C_0(S)$ given by Lemma 1.3, and note that $\{\delta_s : s \in S\}$ is a set of 1-measures for $\mathscr{G}$. (δ_s is the measure for the smallest element G of $\mathscr{G}$ with $s \in G$.) Thus if we define $\phi: \mathscr{G} \to X$ by $\phi(G) = TL1_G$ and $\psi: \mathscr{G} \to A \subset X^*$ by $\psi(G_s) = (\rho^{-1}L^*T^*)^{-1}\delta_s$, $(\phi, \mathscr{G})$ is a $\|T\|$-bounded $\widehat{\omega^\alpha}$-family of functions in X with ρ-measures in B as required.

Conversely, let $(\phi, \mathscr{G})$ be a λ-bounded $\widehat{\omega^\alpha}$-family of functions in X with δ - measures in B. Define $\tilde{\phi}: \overline{\text{span}}\{1_G : G \in \mathscr{G}\} \to X$ by $\tilde{\phi}(1_G) = \phi(G)$ and extend linearly. By definition $\|\tilde{\phi}\| \leqslant \lambda$. By Lemma 1.4, $Z = \overline{\text{span}}\{1_G : G \in \mathscr{G}\}$ is isometric to $C_0(\omega^{\omega^\alpha})$ and moreover there is a Boolean algebra isomorphism ξ of the algebra generated by $\mathscr{G}$ into the subsets of $[1, \omega^{\omega^\alpha})$ which induces the isometry.

Hence we can define an isometry $\tilde{\xi}$: $C_0(\omega^{\omega^\alpha}) \to Z$ by letting $\tilde{\xi}(1_{\xi(G)}) = 1_G$ for all $G \in \mathscr{G}$ and extending linearly. Clearly $\|\tilde{\phi}\tilde{\xi}\| \leqslant \lambda$ and $(\tilde{\phi}\tilde{\xi})^*(B)$ is a w^* closed convex symmetric subset of $\lambda B_{C_0(\omega^{\omega^\alpha})^*}$. Because $(\phi, \mathscr{G})$ has δ-measures in B, $\mathscr{F} = \{\xi(G) : G \in \mathscr{G}\}$ is an $\widehat{\omega^\alpha}$-family of subsets of $[1, \omega^{\omega^\alpha})$ with δ-measures in $(\tilde{\phi}\tilde{\xi})^*(B)$. Indeed if $\psi: \mathscr{G} \to B$ such that $\tilde{\phi}^*(\psi(G))(G) \geqslant \delta$ and $\{\tilde{\phi}^*[\psi(G)] : G \in \mathscr{G}\}$ is a set of δ-measures for $\mathscr{G}$, $\{\tilde{\xi}^*\tilde{\phi}^*[\psi(G)] : G \in \mathscr{G}\}$ is a set of δ-measures for $\mathscr{F}$. By Theorem 0.2 of [1] there is a subspace Y of $C_0(\omega^{\omega^\alpha})$ such that $\tilde{\phi}\tilde{\xi}|_Y$ is an isomorphism, Y is isometric to $C_0(\omega^{\omega^\alpha})$, and Y is normed by $\{\tilde{\xi}^*\tilde{\phi}^*[\psi(G)] : G \in \mathscr{G}\}$. Hence $\tilde{\phi}\tilde{\xi}(Y)$ is isomorphic to $C_0(\omega^{\omega^\alpha})$ and is normed by B, as claimed.

Remark 1.2. In the case where $X = C(K)$ we were able to obtain a subspace isometric to $C_0(\omega^{\omega^\alpha})$. In this generality it is impossible even to obtain subspaces $1+\varepsilon$ isomorphic to $C_0(\omega^{\omega^\alpha})$ because of the negative solution to the distorted norm problem [14]. It should be noted that as λ and δ tend to one, the product of the norming constant and the distance to $C_0(\omega^{\omega^\alpha})$ tend to one.

COROLLARY 1.6. *Let X be a separable Banach space and let T be a bounded linear operator from X into a Banach space Y. Then there is a subspace Z of X such that Z is isomorphic to $C_0(\omega^{\omega^\alpha})$ and $T|_Z$ is an isomorphism if and only if there are constants λ and δ, $0 < \delta, \lambda < \infty$ and a λ-bounded $\widehat{\omega^\alpha}$-family of functions in X with δ-measures in $T^*B_{Y^*}$.*

Remark 1.3. In [1] we were able to conclude from the fact that an operator $T: C(K) \to Y$ was an isomorphism on subspaces uniformly isomorphic to $C_0(\omega^{\alpha_n})$, $\alpha_n \uparrow \omega^\alpha$ that there was a subspace isomorphic to $C_0(\omega^{\omega^\alpha})$ on which T was an isomorphism. No such extension of our results is possible in this generality because X may contain $C_0(\omega^{\alpha_n})$ isometrically but not contain $C_0(\omega^{\omega^\alpha})$, e.g., if $X = (\sum C_0(\omega^{\alpha_n}))_{l_1}$.

The next result will be used in the next section. It clarifies the relationship between $\widehat{\omega^\alpha}$-families of functions and $\widehat{\omega^\alpha}$-families of sets.

PROPOSITION 1.7. *Let K be a compact metric space, let X be a subspace* *of $C(K)$, and let $J: X \to C(K)$ be the inclusion. If B is a w^* closed convex symmetric subset of B_{X^*} and there is a subspace Y of X such that B norms Y and Y is isomorphic to $C_0(\omega^{\omega^\alpha})$ then for any selection $j: B \to B_{C(K)^*}$ such that $J^* \circ j$ is the identity on B there exist an $\varepsilon > 0$ and an $\widetilde{\omega^\alpha}$ family of subsets of K with ε-measures in $\overline{j(B)}^{w^*}$.*

Proof. Let T be an isomorphism of $C_0(\omega^{\omega^\alpha})$ onto Y. As in the proof of Proposition 1.3 we may assume that there is a set $A \subset \overline{j(B)}^{w^*}$ and $a > 0$ such that aT^*J^* is a homeomorphism of A onto the point masses on $[1, \omega^{\omega^\alpha})$ union zero. By Proposition 2.0 of [1] there is an $\widetilde{\omega^\alpha}$-family of subsets of K with ε-measures in A.

Combining Proposition 1.3 with Proposition 1.7 we see that a bounded $\widetilde{\omega^\alpha}$-family of functions with δ-measures always implies the existence of an $\widetilde{\omega^\alpha}$-family of subsets with $\varepsilon(\delta)$-measures. In the next two sections we will investigate the converse of this statement.

2. Application to the Disc Algebra

For our purposes the disc algebra $\mathscr{A}$ will be the subspace of the continuous functions on the boundary of the unit disc D in the complex plane which have extensions to D that are analytic on the interior of D. Delbaen [5] and Kisliakov [12] considered the case of non-weakly compact operators with domain $\mathscr{A}$ and obtained the result that such an operator must be an isomorphism on a copy of c_0. In this section we will extend this result by characterizing $C_0(\omega^{\omega^\alpha})$ preserving operators in terms of $\widetilde{\omega^\alpha}$-families of sets. This improvement of the result of Section 1 is made possible by the following version of the Havin Lemma [10] or [17]. In the sequel $m(\cdot)$ will denote normalized Lebesgue measure on ∂D which we identify with $[0, 2m)$, and $d(\theta_1, \theta_2) = |\theta_1 - \theta_2|$.

LEMMA 2.1. *Let E be a closed subset of ∂D and let $\pi \geqslant \varepsilon > m(E) > 0$. There exists a function $f \in \mathscr{A}$ and a constant B independent of ε and $m(E)$ such that*

(i) $\|f\|_\infty < 1$,

(ii) $\sup\{|f(\theta) - 1|: \theta \in E\} < B(m(E)/\varepsilon)^{1/2}$,

(iii) $|f(\theta)| < B(m(E)/\varepsilon)^{1/2}$ *for all θ for which*

$$2\varepsilon < d(\theta, E) = \inf\{d(\theta, \theta_1): \theta_1 \in E\}.$$

Proof. Let F be a finite union of closed intervals in ∂D such that

$E \subset \operatorname{Int}(F)$, $m(F) < 2m(E)$ and for all $\theta \in F$, $d(\theta, E) < \varepsilon$. Let $\delta = \varepsilon^{-1} m(E)$ and let w be a C^∞-function from ∂D into $[-\delta^{-1/2}, -\delta^{1/2}]$ such that

$$\begin{aligned} w(\theta) &= -\delta^{-1/2} && \text{if } \theta \in E \\ &= -\delta^{1/2} && \text{if } \theta \in F^c. \end{aligned}$$

By the Poisson integral formuula w extends to be a harmonic nonzero function on D with harmonic conjugate

$$v(r, \theta) = \pi^{-1} \int_0^{2\pi} \frac{r \sin(\theta - \phi)\, w(\phi)}{1 - 2r\cos(\theta - \phi) + r^2}\, d\phi, \qquad 0 \leqslant r < 1.$$

Let $v(\theta) = \lim_{r \to 1} v(r, \theta)$ and $h = w + iv$. By [7, p. 83], v is continuous and because $w < 0$ on D, h^{-1} is analytic on the interior of D and continuous on ∂D, i.e., $h^{-1} \in \mathscr{A}$. Let $f = \exp(h^{-1})$.

For (i) observe that $\operatorname{Re} h^{-1} = w(w^2 + \gamma^2)^{-1} < 0$ and thus $|f(\theta)| < 1$ for all θ. If $\theta \in E$, $|f(\theta) - 1| = |\exp(h^{-1}) - 1| \leqslant |h^{-1}| \exp |h^{-1}| \leqslant |h^{-1}| \exp(1) \leqslant |w|^{-1} \exp(1) = \delta^{1/2} \exp(1)$.

Property (iii) requires that $|v(\theta)|$ be small for θ satisfying $d(\theta, E) > 2\varepsilon$.

$$\begin{aligned} |v(\theta)| &= \lim_{r \to 1} \pi^{-1} \left| \int_0^{2\pi} \frac{r \sin(\theta - \phi)\, w(\phi)}{1 - 2r\cos(\theta - \phi) + r^2}\, d\phi \right| \\ &= \lim_{r \to 1} \pi^{-1} \left| \int_0^{\pi} \frac{r \sin \psi [w(\theta - \psi) - w(\theta + \psi)]}{1 - 2r\cos\psi + r^2}\, d\psi \right| \end{aligned}$$

by the change of variables

$$\begin{aligned} \psi &= -(\theta - \phi) && \theta \leqslant \phi \leqslant \theta + \pi \\ &= \theta - \phi && \theta - \pi \leqslant \phi \leqslant \theta. \end{aligned}$$

If both $\theta - \psi$ and $\theta + \psi$ are in F^c, $w(\theta - \psi) = w(\theta + \psi) = -\delta^{1/2}$. Hence

$$|v(\theta)| = \lim_{r \to 1} \pi^{-1} \left| \int_A \frac{r \sin \psi [w(\theta - \psi) - w(\theta + \psi]}{1 - 2r\cos\psi + r^2}\, d\psi \right|,$$

where $A = \{\psi\colon\ \theta + \psi \in F \text{ or } \theta - \psi \in F,\ 0 \leqslant \psi \leqslant \pi\}$. If $d(\theta, E) > 2\varepsilon$, $\psi \in A$ only if $\psi > \varepsilon$, and thus

$$\begin{aligned} |v(\theta)| &\leqslant 4m(F)[\delta^{-1/2} - \delta^{1/2}] \max \left\{ \left(\frac{\sin\psi}{2 - 2\cos\psi} \right) : \psi \in A \right\} \\ &\leqslant 8m(E)\, \delta^{-1/2} \frac{\sin\varepsilon}{2(1 - \cos\varepsilon)} \leqslant 8m(E)\, \delta^{-1/2} \varepsilon^{-1} \leqslant 8\delta^{1/2}. \end{aligned}$$

Therefore

$$|f(\theta)| \leqslant \exp\left(\frac{w}{w^2+v^2}\right) \leqslant \exp\left(\frac{-\delta^{1/2}}{\delta+64\delta}\right) = \exp\left(\frac{-\delta^{-1/2}}{65}\right) \leqslant 65\delta^{1/2}.$$

Thus we can let $B = 65$.

Remark 2.1. An alternate proof of this lemma can be based on the argument of Garnett [9].

The next lemma is a minor modification of Lemma 2.1 which will allow us to construct subspaces of $\mathscr{A}$ isometric to $C_0(\omega^{\omega^\alpha})$ by using the Rudin–Carleson Theorem [8] and [17].

LEMMA 2.2. *Let E and E' be closed subsets of ∂D and let $\pi > \varepsilon > m(E) > 0$. Suppose that $d(E, E') > 2\varepsilon$ and $m(E') = 0$. Then there exists a function $f \in \mathscr{A}$ such that*

(i) $\|f\|_\infty \leqslant 1$,
(ii) $\sup_{\theta\in E} |f(\theta) - 1| < B(m(E)/\varepsilon)^{1/2}$,
(iii) $|f(\theta)| < 2B(m(E)/\varepsilon)^{1/2}$ *if* $d(\theta, E) > 2\varepsilon$,
(iv) $f(\theta) = 0$ *for all* $\theta \in E'$.

Proof. Let g be the function given by Lemma 2.1 Let h be a positive continuous function on ∂D such that

(a) $h(\theta) \leqslant 1 - |g(\theta)|$ for all $\theta \in \partial D$,
(b) $h(\theta) \leqslant B(m(E)/\varepsilon)^{1/2} - |1 - g(\theta)|$ for $\theta \in E$,
(c) $h(\theta) \geqslant |g(\theta)|$ for $\theta \in E'$,
(d) $h(\theta) \leqslant B(m(E)/\varepsilon)^{1/2}$ for $d(\theta, E) > 2\varepsilon$.

By the Rudin–Carleson Theorem there is a function $g_1 \in \mathscr{A}$ such that $g_1|_{E'} = g|_{E'}$ and $|g_1(\theta)| \leqslant h(\theta)$ for all $\theta \in \partial D$. Clearly $f = g - g_1$ is the required function.

We can now show that there are sufficiently many functions in $\mathscr{A}$ so that we can use $\widehat{\omega^\alpha}$-families of *sets* instead of functions to characterize $C_0(\omega^{\omega^\alpha})$ norming subsets of $\mathscr{A}^*$.

PROPOSITION 2.3. *Let $\mathscr{G} = \{G_\alpha : \alpha < \omega^{2\beta}\}$ be a $\widehat{2\beta}$-family of open subsets of ∂D, let $\{F_\alpha : \alpha < \omega^{2\beta}\}$ be a family of closed subsets of ∂D such that $\overline{G_\alpha \supset F_\alpha \supset \{G_\beta : G_\beta \subsetneq G_\alpha\}}$, and let E' be a closed subset of $\overline{\bigcup \{G_\alpha : \alpha < \omega^{2\beta}\}^c}$ with $m(E') = 0$. Then for every $\delta > 0$ there is a $(3+\delta)$-bounded β-family $(\phi, \mathscr{F})$ of functions in $\mathscr{A}$ such that*

(i) $\mathscr{F} \subset \mathscr{G}$,

(ii) $|\phi(G_\alpha)(\theta) - 1| < \delta$ *for all* $\theta \in F_\alpha$ *and* $G_\alpha \in \mathscr{F}$,

(iii) $\sum_{G \in \mathscr{F}} |\phi(G)(\theta)| < \delta$ *for all* $\theta \in \overline{(\bigcup \{G_\alpha : \alpha < \omega^{2\beta}\})^c}$ *and* $\sum_{G \in \mathscr{F}} |\phi(G)(\theta)|$ *is continuous in* $\overline{\bigcup \{G_\alpha : \alpha < \omega^{2\beta}\}^c}$,

(iv) $\sum_{G \cap H = \varnothing, G \in \mathscr{F}} |\phi(G)(\theta)| < \delta$ *for all* $\theta \in H$, $H \in \mathscr{F}$,

(v) $\sum_{G \in \mathscr{F}} |\phi(G)(\theta) - \sum_{H \sqsubset G} \phi(H)(\theta)| < 1 + \delta$ *for all*

$$\theta \in \bigcup \{G_\alpha : \alpha < \omega^{2\beta}\} - \bigcup \{F_\alpha : G_\alpha \in \mathscr{F}\},$$

(vi) $\phi(G)(\theta) = 0$ *for all* $\theta \in E'$, $G \in \mathscr{F}$. *Moreover, if* $\{\mu_\alpha : \alpha < \omega^{2\beta}\}$ *is a set of* ε*-measures for* $\mathscr{G}$ *then* $\mathscr{F}$ *can be chosen such that* $\{\mu_\alpha : G_\alpha \in \mathscr{F}\}$ *is a set of* ε*-measures for* $\mathscr{F}$.

Proof. We will construct the $\hat{\beta}$-families by induction. If $\beta = 1$, let $\delta_n \downarrow 0$ such that $\sum_{n=1}^\infty \delta_n < \delta$. For each n, $\{G_{\omega n + k} : k \in \mathbb{N}\}$ is a sequence of disjoint subsets of $G_{\omega(n+1)}$. Because $\overline{\bigcup \{G_{\omega n + k} : k \in \mathbb{N}\}} \subset G_{\omega(n+1)}$, $\inf\{d(G^c_{\omega(n+1)}, G_{\omega n + k}) : k \in \mathbb{N}\} = a_n > 0$. Thus, for each n there is an integer $k(n) \in \mathbb{N}$ such that $B(3a_n^{-1} m(G_{\omega n + k(n)}))^{1/2} < 4^{-1}\delta_n$. It follows from Lemma 2.2 that for each n there is a function f_n in $\mathscr{A}$ such that

(1) $\|f_n\| \leqslant 1$,

(2) $\sup_{\theta \in F_{\omega n + k(n)}} |f_n(\theta) - 1| < 4^{-1}\delta_n$,

(3) $|f_n(\theta)| < 2^{-1}\delta_n$ if $d(\theta, F_{\omega n + k(n)}) > 2a_n/3$,

(4) $f(\theta) = 0$ for all $\theta \in E'$.

Clearly $\mathscr{F} = \{G_{\omega n + k(n)} : n \in N\}$ is a $\hat{1}$-family and the map $\phi(G_{\omega n + k(n)}) = f_n$ defines an $\hat{1}$-family of functions in $\mathscr{A}$. Because

$$\sum_{n=1}^{\infty} |f_n(\theta)| \leqslant 2^{-1} \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \delta_n + |f_m(\theta)| < 1 + \delta \qquad \text{if} \quad \theta \in G_{\omega m} \tag{2.0}$$

we have a $(1 + \delta)$ bounded $\hat{1}$-family. Properties (i)–(vi) are obvious.

The remainder of the proof is simply to iterate this argument inductively. Assume that the proposition holds for all $\gamma < \beta$. The case where β is a successor and the case where β is a limit ordinal are similar so we will prove only the successor case.

Let $\gamma + 1 = \beta$. By the definition of a $\widehat{2\beta}$ family there is a sequence of disjoint sets $\{G_{\alpha(n)} : \alpha(n),\ n \in \mathbb{N}\} \subset \mathscr{G}_{2\gamma+1}$ and for each n a sequence of disjoint subsets $\{G_{\alpha(n,k)} : \alpha(n,k),\ k \in \mathbb{N}\}$ of $G_{\alpha(n)}$ such that $\mathscr{G}_{n,k} = \{G_\alpha : G_\alpha \subsetneq G_{\alpha(n,k)}\}$ is a $\widehat{2\gamma}$ family for each k. Let $\delta_n > 0$ such that $\sum_{n=1}^\infty \delta_n < 5^{-1}\delta$. By the first step of the induction there is a $(1 + 5^{-1}\delta)$-bounded $\hat{1}$-family of functions in $\mathscr{A}$, $(\phi, \mathscr{F})$ with $\mathscr{F} = \{G_{\alpha(n,k(n))} : n \in \mathbb{N}\}$ such that

(1) $\|\phi(G)\| \leqslant 1$ for all $G \in \mathscr{F}$,

(2) $|\phi(G_{\alpha(n,k(n))})(\theta)-1|<5^{-1}\delta$ for all $\theta\in F_{\alpha(n,k(n))}$, $n\in\mathbb{N}$,

(3) $\sum_{G\in\mathscr{F}}|\phi(G)(\theta)|<5^{-1}\delta$ for all $\theta\in(\bigcup\{G_{\alpha(n)}: n\in\mathbb{N}\})^c$,

(4) $\sum_{n=1,n\neq m}^{\infty}|\phi(G_{\alpha(n,k(n))})(\theta)|<5^{-1}\delta$ for all $\theta\in G_{\alpha(m,k(m))}$,

(5) $\phi(G)(\theta)=0$ for all $\theta\in E'$ and $G\in\mathscr{F}$.

For each n we get by the inductive hypothesis a $(3+\delta_n)$-bounded $\hat{\gamma}$-family of functions $(\phi_n,\mathscr{F}_n)$ such that

(i)′ $\mathscr{F}_n\subset\mathscr{G}_{n,k(n)}$,

(ii)′ $|\phi_n(G_\alpha)(\theta)-1|<\delta_n$ for all $\theta\in F_\alpha$, $G_\alpha\in\mathscr{F}_n$,

(iii)′ $\sum_{G\in\mathscr{F}_n}|\phi_n(G)(\theta)|<\delta_n$ for all $\theta\in(\bigcup\mathscr{G}_{n,k(n)})^c$, and $\sum_{G\in\mathscr{F}_n}|\phi_n(G)(\theta)|$ is continuous there

(iv)′ $\sum_{G\in\mathscr{F}_n,G\cap H=\varnothing}|\phi_n(G)(\theta)|<\delta_n$ for all $\theta\in H$, $H\in\mathscr{F}_n$,

(v)′ $\sum_{G\in\mathscr{F}_n}|\phi_n(G)(\theta)-\sum_{H\sqsupset G}\phi(H)(\theta)|<1+\delta$ for all $\theta\in\bigcup\mathscr{F}_n-\bigcup\{F_\alpha: G_\alpha\in\mathscr{F}_n\}$,

(vi)′ $\phi_n(G)(\theta)=0$ for all $\theta\in E'$, $G\in\mathscr{F}_n$.

Clearly $\mathscr{H}=\mathscr{F}\cup(\bigcup\{\mathscr{F}_n: n\in\mathbb{N}\})$ is a $\widehat{\gamma+1}$-family and thus the mapping $\psi:\mathscr{H}\to\mathscr{A}$ defined by

$$\begin{aligned}\psi(G)&=\phi(G) &&\text{if } G\in\mathscr{F}\\ &=\phi_n(G) &&\text{if } G\in\mathscr{F}_n\end{aligned}$$

defines a $\widehat{\gamma+1}$-family of functions in $\mathscr{A}$. If $\theta\in\bigcup\mathscr{G}_{n,k(n)}$ for some n, $\theta\in(\bigcup\mathscr{G}_{m,k(m)})^c$ for all $m\neq n$ and thus

$$\sum_{G\in\mathscr{F}_m}\left|\psi(G)(\theta)-\sum_{H\sqsupset G}\psi(H)(\theta)\right|<2\delta_m. \tag{2.1}$$

Because $\theta\in G_{\alpha(n,k(n))}$, by (4),

$$\sum_{\substack{m=1\\ m\neq n}}^{\infty}|\psi(G_{\alpha(m,k(m))})(\theta)|<5^{-1}\delta \tag{2.2}$$

and

$$\begin{aligned}&\sum_{\substack{m=1\\ m\neq n}}^{\infty}\left|\psi(G_{\alpha(m,k(m))})(\theta)-\sum_{H\sqsupset G_{\alpha(m,k(m))}}\psi(H)(\theta)\right|\\ &\leqslant\sum_{\substack{m=1\\ m\neq n}}^{\infty}\left[|\psi(G_{\alpha(m,k(m))})(\theta)+\sum_{H\sqsupset G_{\alpha(m,k(m))}}|\psi(H)(\theta)|\right]\\ &<5^{-1}\delta+\sum_{\substack{m=1\\ m\neq n}}^{\infty}\delta_m.\end{aligned} \tag{2.3}$$

If $\theta \in G_{\alpha(n,k(n))} - \bigcup \{F_\alpha : G_\alpha \in \mathscr{F}_n\}$, by (v)$'$,

$$\sum_{G \in \mathscr{F}_n} \left| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \right| < 1 + \delta_n, \tag{2.4}$$

and thus

$$\begin{aligned}
&\sum_{G \in \mathscr{F}_n} \left| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \right| \\
&\quad = \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \left| \psi(G_{\alpha(m,k(m))})(\theta) - \sum_{H \square G_{\alpha(m,k(m))}} \psi(H)(\theta) \right| \\
&\quad + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \sum_{G \in \mathscr{F}_m} \left| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \right| \\
&\quad + \sum_{G \in \mathscr{F}_n} \left| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \right| \\
&\quad + \left| \psi(G_{\alpha(n,k(n))})(\theta) - \sum_{H \square G_{\alpha(n,k(n))}} \psi(H)(\theta) \right| \\
&\quad < \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \delta_m + 5^{-1}\delta + 2 \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \delta_m \\
&\quad + 1 + \delta_n + |\psi(G_{\alpha(n,k(n))})(\theta)| + \sum_{H \square G_{\alpha(n,k(n))}} |\psi(H)(\theta)| \\
&\quad < 1 + 4 \cdot 5^{-1}\delta + 1 + 1 + \delta_n < 3 + \delta,
\end{aligned} \tag{2.5}$$

by (2.1), (2.3), (2.4) and (1). If $\theta \in \bigcup \{F_\alpha : G_\alpha \in \mathscr{F}_n\}$,

$$\sum_{G \in \mathscr{F}_n} \left| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \right| < 3 + \delta_n \tag{2.6}$$

by the definition of a $(3 + \delta_n)$-bounded family and

$$\begin{aligned}
&\left| \psi(G_{\alpha(n,k(n))})(\theta) - \sum_{H \square G_{\alpha(n,k(n))}} \psi(H)(\theta) \right| \\
&\quad \leqslant |\psi(G_{\alpha(n,k(n))})(\theta) - \psi(H_0)(\theta)| + \sum_{\substack{H \square G_{\alpha(n,k(n))} \\ H \neq H_0}} |\psi(H)|\,(\theta) \\
&\quad \leqslant 5^{-1}\delta + \delta_n + \delta_n \qquad \text{by (2), (ii)}' \text{ and (iv)}',
\end{aligned} \tag{2.7}$$

where $H_0 = G_{\alpha_0} \in \mathscr{F}_n$ and $\theta \in F_{\alpha_0}$. Hence

$$\begin{aligned}
&\sum_{G \in \mathscr{H}} \Big| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \Big| \\
&\qquad \leqslant 3 \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \delta_m + 5^{-1}\delta + 3 + \delta_n + 5^{-1}\delta + 2\delta_n \\
&\qquad < 3 + \delta, \qquad \text{for all} \quad \theta \in \bigcup \{F_\alpha : G_\alpha \in \mathscr{F}_n\}. \qquad (2.8)
\end{aligned}$$

If $\theta \in (\bigcup_n (\bigcup \mathscr{G}_{n,k(n)}))^c$, then by (iii)′

$$\sum_{n=1}^{\infty} \sum_{G \in \mathscr{F}_n} |\phi_n(G)(\theta)| < \sum_{n=1}^{\infty} \delta_n < 5^{-1}\delta \tag{2.9}$$

and thus

$$\begin{aligned}
&\sum_{G \in \mathscr{H}} \Big| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \Big| \\
&\qquad = \sum_{G \in \mathscr{F}} \Big| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \Big| \\
&\qquad\quad + \sum_{n=1}^{\infty} \sum_{G \in \mathscr{F}_n} \Big| \psi(G)(\theta) - \sum_{H \square G} \psi(H)(\theta) \Big| \\
&\qquad \leqslant \sum_{G \in \mathscr{F}} |\psi(G)(\theta)| + \sum_{\substack{H \square G \\ G \in \mathscr{F}_n}} |\psi(H)(\theta)| \\
&\qquad\quad + \sum_{n=1}^{\infty} \sum_{G \in \mathscr{F}_n} \Big[|\psi(G)(\theta)| + \sum_{H \square G} |\psi(H)(\theta)| \Big] \\
&\qquad \leqslant 1 + 5^{-1}\delta + 2 \sum_{n=1}^{\infty} \delta_n < 1 + \delta \qquad (2.10)
\end{aligned}$$

by (2.9) and the definition of a $(1 + 5^{-1}\delta)$-bounded family. Inequality (2.10) proves (v) and with (2.5) and (2.8) shows that $(\psi, \mathscr{H})$ is a $(3 + \delta)$-bounded family of functions in $\mathscr{A}$; (i), (ii) and (vi) are obvious.

If $\theta \in (\bigcup \{G_\alpha : \alpha < \omega^{2\beta}\})^c$, (iii)′ and (2.0) give (iii). For (iv) and $H \subset G_{\alpha(n,k(n))}$,

$$\begin{aligned}
\sum_{\substack{G \in \mathscr{H} \\ G \cap H = \varnothing}} |\psi(G)(\theta)| &\leqslant \sum_{\substack{m=1 \\ m \neq n}}^{\infty} |\psi(G_{\alpha(m,k(m))})(\theta)| + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \sum_{G \in \mathscr{F}_m} |\psi(G)(\theta)| \\
&\quad + \sum_{\substack{G \in \mathscr{F}_n \\ G \cap H = \varnothing}} \psi(G)(\theta) < \delta/5 + \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \delta_m + \delta_n < \delta
\end{aligned}$$

by (4), (iii)′ and (iv)′. The "moreover" assertion is obvious.

In the following if $\mu \in C(\partial D)^*$, $\|\mu\|$ will denote the usual norm and $\|\mu\|_{\mathscr{A}^*}$ will be the norm of μ as an element of $\mathscr{A}^*$.

THEOREM 2.4. *Let $\mathscr{G} = \{G_\gamma : \gamma < \omega^{\omega^\alpha}\}$ be an $\widehat{\omega^\alpha}$-family of open subsets of ∂D with ε-measures $\{\mu_\gamma : \gamma < \omega^{\omega^\alpha}\}$ in $B_{C(\partial D)^*}$ such that $\|\mu_\gamma\|_{\mathscr{A}^*} \geqslant (1-\delta)\|\mu_\gamma\|$ for some δ independent of γ, $\delta < \varepsilon$. Then there is a subspace Y of $\mathscr{A}$ such that Y is isometric to $C_0(\omega^{\omega^\alpha})$ and Y is normed by $\{\mu_\gamma : \gamma < \omega^{\omega^\alpha}\}$.*

Proof. For $\alpha = 0$, this is essentially the result of Delbaen [5] and Kisliakov [12] except that they obtain a subspace of $\mathscr{A}$ isomorphic to c_0. To find a subspace isometric to c_0 one need only modify their construction in a way similar to that of our proof of Lemma 2.2 and use the Rudin–Carleson Theorem as we do below.

Let $\alpha \geqslant 1$. We may assume that there is a closed set E' such that $m(E') = 0$, $(E')^{(\omega^\alpha)} \neq \varnothing$, and there is an open set $\mathscr{O}$ such that $E' \subset \mathscr{O} \subset \bigcup \{G_\gamma : \gamma < \omega^{\omega^\alpha}\}^c$ and $|\mu_\gamma|(\mathscr{O}) < (\varepsilon - \delta)\, 8^{-1}$. Indeed, let $G_{\gamma(1)}, \ldots, G_{\gamma(n)}$ be maximal sets in $\mathscr{G}$, where $n > 8(\varepsilon - \delta)^{-1}$. Then, for some i, $1 \leqslant i \leqslant n$, $A_i = \{\tau : |\mu_\tau|(G_{\gamma(i)}) < 8^{-1}(\varepsilon - \delta)\}$ contains a set homeomorphic to $[1, \omega^{\omega^\alpha})$, by (an easy modification of) Lemma 2.4 of [1], and by Lemma 0.1 we can find an $\widehat{\omega^\alpha}$-family contained in $\mathscr{G}$ with ε-measures in A_i. Because $G_{\gamma(i)}$ is open we can choose the set $E' \subset G_{\gamma(i)}$ and use our new $\widehat{\omega^\alpha}$-family in place of $\mathscr{G}$.

For any $\rho > 0$, we have, by Proposition 2.3, a $(3+\rho)$-bounded $\widehat{\omega^\alpha}$-family $(\phi, \mathscr{H})$ of functions in $\mathscr{A}$ satisfying (i)–(vi), where for each γ, F_γ is a closed subset of G_γ such that $|\mu_\gamma|(F_\gamma) > |\mu_\gamma|(G_\gamma) - \rho$ and $F_\gamma \supset \bigcup \{G_\tau : G_\tau \subsetneq G_\gamma\}$. By the definition of an $\widehat{\omega^\alpha}$-family with measures there exist a sequence $\alpha_n \uparrow \omega^\alpha$ and a sequence of disjoint sets $\{H_n : n \in \mathbb{N}\}$ in $\mathscr{H}$ such that $\mathscr{H}_n = \{H : H \in \mathscr{H},\ H \subset H_n\}$ is an α_n-family and $\{\mu_\gamma : G_\gamma \in \mathscr{H}_n\}$ is homeomorphic to $[1, \omega^{\alpha_n}]$. By Lemma 2.5 of [1] for each $k \in \mathbb{N}$ there is an $n = n(k)$ and a closed subset B_n of $\{\mu_\gamma : G_\gamma \in \mathscr{H}_n\}$ such that B_n is homeomorphic to $[1, \omega^{\alpha_k}]$ and for all $\mu, \nu \in B_n$, $|\|\mu\| - \|\nu\|| < \rho$. By Lemma 0.1 we can find an ω^{α_k}-family $\mathscr{G}_k \subset \mathscr{H}_n$ with measures in B_n. Note that we still have that $(\phi|_{\bigcup \mathscr{G}_k}, \bigcup \mathscr{G}_k)$ is a $(3+\rho)$-bounded $\widehat{\omega^\alpha}$-family of functions in $\mathscr{A}$ satisfying (i)–(vi). Thus to summarize:

There is an $\widehat{\omega^\alpha}$-family $\mathscr{H}' = \{H_\gamma : \gamma < \omega^{\omega^\alpha}\} \subset \mathscr{G}$ with ε-measures $\{\nu_\gamma : \gamma < \omega^{\omega^\alpha}\} \subset \{\mu_\gamma : \gamma < \omega^{\omega^\alpha}\}$ such that if $\gamma, \gamma' \in (\omega^{\alpha_{n-1}}, \omega^{\alpha_n}]$, $|\|\nu_\gamma\| - \|\nu_{\gamma'}\|| < \rho$, $n \in \mathbb{N}$. Further there is a function $\phi : \mathscr{H}' \to \mathscr{A}$ defining a $(3+\rho)$-bounded $\widehat{\omega^\alpha}$-family and closed sets $\{F'_\gamma : \gamma < \omega^{\omega^\alpha}\}$ such that $|\nu_\gamma|(F'_\gamma) > \varepsilon - \rho$ and $F'_\gamma \supset \bigcup \{H_\tau : H_\tau \subsetneq H_\gamma\}$ and satisfying (i)–(vi) of Proposition 2.3.

For each n choose $f_n \in \mathscr{A}$ such that $\|f_n\| \leqslant 1$ and $\langle \nu_{\omega^{\alpha_n}}, f_n \rangle > \|\nu_{\omega^{\alpha_n}}\|_{\mathscr{A}^*} - \rho$. By replacing $(\omega^{\alpha_{n-1}}, \omega^{\alpha_n}]$ by a suitable neighborhood of ω^{α_n} we may assume that $\langle \nu_\gamma, f_n \rangle > \|\nu_{\omega^{\alpha_n}}\|_{\mathscr{A}^*} - \rho$ for all $\gamma \in (\omega^{\alpha_{n-1}}, \omega^{\alpha_n}]$. Define an $\widehat{\omega^\alpha}$-family by $\phi'(H_\gamma) = \phi(H_\gamma) \cdot f_n$ for $\gamma \in (\omega^{\alpha_{n-1}}, \omega^{\alpha_n}]$, $n \in \mathbb{N}$ and measures for $(\phi', \mathscr{H}')$ by $\psi(H_\gamma) = \nu_\gamma$.

Clearly $(\phi', \mathscr{H}')$ is a $(3+\rho)$-bounded $\widehat{\omega^\alpha}$-family of functions in $\mathscr{A}$. Observe that for $\gamma \in (\omega^{\alpha_{n-1}}, \omega^{\alpha_n}]$,

$$\begin{aligned}
\langle \phi'^* \psi(H_\gamma), 1_{H_\gamma} \rangle &= \langle \psi(H_\gamma), \phi'(H_\gamma) \rangle = \langle \nu_\gamma, \phi(H_\gamma) \cdot f_n \rangle \\
&= \langle \nu_\gamma, f_n \rangle - \langle \nu_\gamma, (1 - \phi(H_\gamma)) \cdot f_n \rangle \\
&\geqslant \| \nu_{\omega^{\alpha_n}} \|_{\mathscr{A}^*} - \rho - (1+\rho) \, |\nu_\gamma| \, (H_\gamma^c) \\
&\quad - 2 \, |\nu_\gamma| \, (H_\gamma - F'_\gamma) - \rho \, |\nu_\gamma| \, (F'_\gamma) \\
&\geqslant (1-\delta) \, \| \nu_{\omega^{\alpha_n}} \| - 3\rho \\
&\quad - (1+\rho) \, |\nu_\gamma| \, ((F'_\gamma)^c) - \rho \, |\nu_\gamma| \, (F'_\gamma) \\
&> (1-\delta) \, \| \nu_\gamma \| - (1-\delta)\,\rho - 3\rho \\
&\quad - (1+\rho) \, |\nu_\gamma| \, ((F'_\gamma)^c) - \rho \, |\nu_\gamma| \, (F'_\gamma) \\
&\geqslant |\nu_\gamma| \, (F'_\gamma) - \delta \, \| \nu_\gamma \| - (1-\delta)\,\rho - 4\rho \\
&\geqslant (\varepsilon - \rho) - \delta - (1-\delta)\,\rho - 4\rho.
\end{aligned}$$

Hence for any $\tau > 0$ if ρ is sufficiently small, $\langle \phi'^* \psi(H_\gamma), 1_{H_\gamma} \rangle > \varepsilon - \delta - \tau$. Thus $(\phi', \mathscr{H}')$ is a $(3+\rho)$-bounded $\widehat{\omega^\alpha}$-family of functions in $\mathscr{A}$ with $(\varepsilon - \delta - \tau)$-measures.

By the proof of Proposition 1.3 there is a subspace Y of $\mathscr{A}$ such that Y is isomorphic to $C_0(\omega^{\omega^\alpha})$ and normed by $\{\mu_\gamma : \gamma < \omega^{\omega^\alpha}\}$. To obtain a subspace isometric to $C_0(\omega^{\omega^\alpha})$ which is normed by $\{\mu_\gamma : \gamma < \omega^{\omega^\alpha}\}$ we resort to the following subterfuge:

Let $a(\theta)$ be a positive continuous function on ∂D such that

(a) $a(\theta) \leqslant 1 - \sum_{\gamma < \omega^{\omega^\alpha}} |\phi(G_\gamma)(\theta)|$ for θ in $\mathscr{O}$,

(b) $a(\theta) < \rho(3+2\rho)^{-1}$ for $\theta \in \mathscr{O}^c$,

(c) $a(\theta) = 1$ for $\theta \in E'$.

By the Rudin–Carleson Theorem there is a subspace Z of $\mathscr{A}$ such that the restriction of Z to E' is an isometry onto $C(E')$ and for all $z \in Z$, $|z(\theta)| \leqslant a(\theta) \, \|z\|$ for all $\theta \in \partial D$. Because $(E')^{(\omega^\alpha)} \neq \varnothing$ there is a subspace of $C(E')$ isometric to $C_0(\omega^{\omega^\alpha})$ and hence a 1-bounded $\widehat{\omega^\alpha}$-family of functions in Z, $(\phi'', \mathscr{H}')$, with 1 measures, i.e., $\overline{\mathrm{sp}}\, \phi''(\mathscr{H}')$ is isometric to $C_0(\omega^{\omega^\alpha})$. Define a new $\widehat{\omega^\alpha}$-family of functions by $\xi(H_\gamma) = \phi''(H_\gamma) + (3+2\rho)^{-1} \phi'(H_\gamma)$ for all $\gamma < \omega^{\omega^\alpha}$. Note that

$$\begin{aligned}
&\sum_{H \in \mathscr{H}'} \Big| \xi(H)(\theta) - \sum_{I \sqsupset H} \xi(I)(\theta) \Big| \\
&\quad \leqslant \sum_{H \in \mathscr{H}'} \Big| \phi''(H)(\theta) - \sum_{I \sqsupset H} \phi''(I)(\theta) \Big|
\end{aligned}$$

$$+ (3+2\rho)^{-1} \sum_{H \in \mathscr{H}'} \left| \phi'(H)(\theta) - \sum_{I \sqsupset H} \phi'(I)(\theta) \right|$$
$$\leqslant a(\theta) + (3+2\rho)^{-1} 2 \sum_{\gamma < \omega^{\omega^\alpha}} |\phi(G_\gamma)(\theta)| \, 1_{\mathscr{O}}(\theta)$$
$$+ (3+2\rho)^{-1} (3+\rho) \, 1_{\mathscr{O}^c}(\theta) \leqslant 1$$

by the definition of $a(\theta)$. Thus $(\xi, \mathscr{H}')$ is a 1-bounded $\widehat{\omega}^\alpha$-family of functions in $\mathscr{A}$ and because $\xi(H_\gamma)|_{E'} = \phi''(H_\gamma)|_{E'}$, $\overline{\mathrm{sp}}\{\xi(H_\gamma)\colon \gamma < \omega^{\omega^\alpha}\}$ is isometric to $C_0(\omega^{\omega^\alpha})$. Finally

$$\begin{aligned} \langle \psi(H_\gamma), \xi(H_\gamma)\rangle &= \langle \psi(H_\gamma), \phi''(H_\gamma)\rangle + (3+2\rho)^{-1} \langle \psi(H_\gamma), \phi'(H_\gamma)\rangle \\ &\geqslant (\varepsilon - \delta - \tau)(3+2\rho)^{-1} - \langle |\psi(H_\gamma)|, a(\theta)\rangle \\ &\geqslant (\varepsilon - \delta - \tau)(3+2\rho)^{-1} - |\psi(H_\gamma)|(\mathscr{O}) - \rho(3+2\rho)^{-1} \|\psi(H_\gamma)\| \\ &\geqslant (\varepsilon - \delta - \tau)(3+2\rho)^{-1} - (\varepsilon - \delta)\, 8^{-1} - \rho(3+2\rho)^{-1}. \end{aligned}$$

Thus for ρ sufficiently small ψ defines $(\varepsilon - \delta)\, 6^{-1}$ measures for $(\xi, \mathscr{H}')$. Thus there is a subspace W of $\mathscr{A}$ such that W is isometric to $C_0(\omega^{\omega^\alpha})$ and is normed by $\{\mu_\gamma\colon \gamma < \omega^{\omega^\alpha}\}$.

Remark 2.1. In the above argument we obtain a subspace isometric to $C_0(\omega^{\omega^\alpha})$ only by sacrificing the norming constant. The product of the norming constant and the isomorphism constant is always greater than 3 in this construction.

PROBLEM. Let B be a w^* closed convex symmetric subset of $B_{\mathscr{A}^*}$ and let B norm a subspace Y of $\mathscr{A}$ such that Y is isomorphic to $C_0(\omega^{\omega^\alpha})$. Given $\varepsilon > 0$, is there a subspace Z of $\mathscr{A}$ such that $d(Z, C_0(\omega^{\omega^\alpha})) < 1 + \varepsilon$ and $\sup\{\rho\colon \sup\{|b(z)|\colon b \in B\} \geqslant \rho \|z\| \text{ for all } z \in Z\} \geqslant (1-\varepsilon) \sup\{\rho\colon \sup\{|b(y)|\colon b \in B\} \geqslant \rho \|y\| \text{ for all } y \in Y$?

COROLLARY 2.5. *Let B be a w^* closed convex symmetric subset of $B_{\mathscr{A}^*}$ and let $J\colon \mathscr{A} \to C(\partial D)$ be the inclusion. A necessary and sufficient condition that B norm a subspace Y of $\mathscr{A}$ such that Y is isomorphic to $C_0(\omega^{\omega^\alpha})$ is that there exist constants ε and δ, $\varepsilon > \delta > 0$, a function $j\colon B \to B_{C(\partial D)^*}$ such that $J^* \circ j = id_B$ and $\|b\| \geqslant (1-\delta)\|j(b)\|$ for all $b \in B$, and an ω^α-family of subsets of ∂D with ε-measures in $j(B)$.*

Proof. Use Proposition 1.7 and Theorem 2.4.

3. Subspaces of $C(K)$ with Small Annihilator

In this section we consider an arbitrary subspace X of $C(K)$, K compact metric, such that $X^{\perp}$ is separable and obtain results similar to those of the previous section under certain conditions on the size of $X^{\perp}$. The main difficulty is to find a suitable replacement for the Havin Lemma. The first step is the following consequence of the Hahn–Banach Theorem:

LEMMA 3.1. *Let X be a subspace of $C(K)$ for some compact Hausdorff space K and let E and F be disjoint closed subsets of K. Let $f \in C(K)$ such that $|f(k)| = 1$ for all $k \in E$, $f(k) = 0$ for all $k \in F$. Suppose that there is an $\eta > 0$ such that for all $\mu \in X^{\perp}$, $\|\mu\| \leqslant 1$, there is a function $\phi \in C_{\mathbb{R}}(K) = \{g \in C(K)\colon g(K) \subset \mathbb{R}\}$, $1 \geqslant \phi \geqslant 1_E$, such that $|\langle \mu, \phi f\rangle| < \eta/2$. Then there is a function $h \in X$ such that $|h(k) - f(k)| < \eta$ for all $k \in E \cup F$ and $\|h\| \leqslant 3$.*

Proof. Suppose that the lemma is false. For each $\tau > 0$ let $W_\tau = \overline{\mathrm{co}}((\tau B_{C(K)}) + f) \cup \{\phi f\colon \phi \in C_{\mathbb{R}}(K) \text{ and } 1 \geqslant \phi \geqslant 1_E\}$. Let B be the circled hull of $X + (W_\tau - f)$ and let $\rho(h) = \inf\{t > 0\colon t^{-1}h \in B\}$, the Minkowski functional for B. Note that because $B \supset \tau B_{C(K)}$, $\rho(h) \leqslant \tau^{-1}\|h\|$, for all h.

We claim that $\rho(f) \geqslant \eta\tau^{-1}$. If $sf \in B$, for some s, $s^{-1} < \eta\tau^{-1}$, then $sf = \lim_{i\to\infty} [x_i + \sum_{j=1}^{n(i)} \gamma_j(w_j - f)]$, where $x_i \in X$, $\sum_{j=1}^{n(i)} |\gamma_j| = 1$, and $\{w_j\} \subset W_\tau$ for each i. Thus

$$\left\| x_i' - \left[f - s^{-1} \sum_{j=1}^{n(i)} \gamma_j(w_j - f) \right] \right\| \to 0, \qquad \text{where} \quad x_i' = s^{-1}x_i \in X.$$

We will show that for large i, x_i' satisfies the conclusion of the lemma—contradicting our assumption.

Let $y_i = f - s^{-1} \sum_{j=1}^{n(i)} \gamma_j(w_j - f)$, then for $k \in E \cup F$

$$|(y_i - f)(k)| \leqslant s^{-1} \sum_{j=1}^{n(i)} |\gamma_j|\, |(w_j - f)(k)| \leqslant s^{-1}\tau < \eta.$$

Also

$$\begin{aligned} \|y_i\| &\leqslant \|f\| + s^{-1} \sum_{j=1}^{n(i)} |\gamma_j|\, \|w_j - f\| \\ &\leqslant \|f\| + s^{-1} \sup\{\|w_j - f\|\colon 1 \leqslant j \leqslant n(i)\}. \end{aligned}$$

Because $w_j \in W_\tau$, $w_j = \lambda(\tau h + f) + (1 - \lambda)\phi f$ for some λ, $0 \leqslant \lambda \leqslant 1$, $h \in B_{C(K)}$, and ϕ, $1 \geqslant \phi \geqslant 1_E$. Thus $\|w_j - f\| \leqslant \lambda\tau\|h\| + (1-\lambda)\|(\phi - 1)f\| \leqslant 1$ and hence $\|y_i\| \leqslant 1 + s^{-1} < 1 + \eta\tau^{-1}$. In particular if $\tau = \eta/2$, $\|y_i\| < 3$. It follows then that for sufficiently large i, $\|x_i'\| < 3$ and for all $k \in E \cup F$, $|(x_i - f)(k)| < \eta$, a contradiction.

By the Hahn–Banach Theorem there is a linear functional ν on $C(K)$ such that $\nu(f)=\rho(f)$ and $|\nu(h)|\leqslant\rho(h)$ for all $h\in C(K)$. Note that $|\nu(x)|\leqslant\rho(x)=0$ for all $x\in X$ and hence $\nu\in X^{\perp}$. Because $\rho(h)\leqslant\tau^{-1}\|h\|$, $\|\nu\|\leqslant\tau^{-1}$. If $h\in W_\tau$, $\rho(h-f)\leqslant 1$, and thus $|\nu(h)|\geqslant|\nu(f)|-\rho(h-f)\geqslant\eta\tau^{-1}-1$. Letting $\mu=\|\nu\|^{-1}\nu$ we have that $|\mu(h)|\geqslant\eta-\tau=\eta/2$, if $\tau=\eta/2$. This contradicts our hypothesis, proving the lemma.

DEFINITION. Let B be a subset of $C(K)^*$ for some compact metric space K and let $\varepsilon>0$. Let $\mathscr{W}(\varepsilon,B)=\sup\{\alpha\colon\exists$ an α-family of subsets of K with ε-measures in $B\}$.

It follows from the proof of Theorem 0.2 of [1] that if $\mathscr{W}(\varepsilon,B)=\omega^\beta$ for some $\beta<\omega_1$, and if $\varepsilon'<\varepsilon$, there is an $\widehat{\omega^\beta}$-family of subsets of K with ε'-measures in B.

The next lemma will be our replacement for Lemma 2.1.

LEMMA 3.2. *Let X be a subspace of $C(K)$ such that $\mathscr{W}(\varepsilon,B_{X^\perp})<\alpha$ for some ordinal $\alpha<\omega_1$. Then if $\mathscr{G}=\{G_\gamma\colon\gamma\leqslant\omega^\alpha\}$ is an α-family of subsets of K, $\mathscr{F}=\{F_\gamma\colon\gamma\leqslant\omega^\alpha\}$ is a family of closed subsets of K such that $\bigcup\{G_\tau\colon G_\tau\subsetneq G_\gamma\}\subset F_\gamma\subset G_\gamma$, for all $\gamma\leqslant\omega^\alpha$,* and *$f\in C(K)$ such that $\|f\|=1$, $|f(k)|=1$ for all $k\in F_{\omega^\alpha}$ and $f(k)=0$ for all $k\in G^c_{\omega^\alpha}$, then there is a function $h\in X$, $\|h\|\leqslant 3$, and a set $G_\gamma\in\mathscr{G}$ such that $|h(k)-f(k)|<2\varepsilon$ for all $k\in F_\gamma\cup G^c_{\omega^\alpha}$.*

Proof. Suppose that no such function h exists. By Lemma 3.1 for each $\gamma<\omega^\alpha$ there is a measure $\mu_\gamma\in B_{X^\perp}$ such that $|\langle\mu_\gamma,\phi f\rangle|\geqslant\varepsilon$ for all $\phi\in C_{\mathbb{R}}(K)$, $1\geqslant\phi\geqslant 1_{F_\gamma}$. Observe that if $F_\gamma\subset F_{\gamma'}$, $\{\phi f\colon 1\geqslant\phi\geqslant 1_{F_\gamma}\}\supset\{\phi f\colon 1\geqslant\phi\geqslant 1_{F_{\gamma'}}\}$ and thus, if $\mu_{\gamma(n)}\to^{w^*}\mu$ and $\bigcup\{G_{\gamma(n)}\colon n\in\mathbb{N}\}\subset F_\gamma$, then $\langle\mu,\phi f\rangle\geqslant\varepsilon$ for all ϕ, $1\geqslant\phi\geqslant 1_{F_\gamma}$. In particular $\langle\mu,1_{F_\gamma}f\rangle\geqslant\varepsilon$ and $|\mu|(G_\gamma)\geqslant\varepsilon$. A simple transfinite induction argument shows that $\mathscr{H}=\{G_\gamma\colon\gamma<\omega^\alpha\}$ contains an $\hat{\alpha}$-family with ε-measures in $\overline{\{\mu_\gamma\colon\gamma<\omega^\alpha\}}^{w^*}\subset B_{X^\perp}$, contradicting the assumption that $\mathscr{W}(\varepsilon,B_{X^\perp})<\alpha$.

The next result is the analog of Proposition 2.3.

PROPOSITION 3.3. *Let X be a subspace of $C(K)$ and $\alpha<\omega_1$ such that $\mathscr{W}(\varepsilon,B_{X^\perp})<\alpha$ for every $\varepsilon>0$. Let $\mathscr{G}=\{G_\gamma\colon\gamma<\omega^{\alpha\beta}\}$ be an $\widehat{\alpha\beta}$-family of open subsets of K, let $\mathscr{F}=\{F_\gamma\colon\gamma<\omega^{\alpha\beta}\}$ be a family of closed subsets of K such that $\bigcup\{G_\tau\colon G_\tau\subsetneq G_\gamma\}\subset F_\gamma\subset G_\gamma$ for all $\gamma<\omega^{\alpha\beta}$, and let $f\in C(K)$ such that $|f(k)|=1$ for all $k\in\bigcup\mathscr{G}$. Then for every $\delta>0$ there is a $(7+\delta)$-bounded $\hat{\beta}$-family of functions in X, $(\psi,\mathscr{H})$, such that*

(i) $\mathscr{H}\subset\mathscr{G}$.

(ii) $|\psi(G_\gamma)(k)-f(k)|<\delta$ *for all* $k\in F_\gamma$,

$$|\psi(G_\gamma)(k)|<\delta\qquad\textit{for all}\quad k\in G^c_\gamma,\quad\textit{if}\quad G_\gamma\in\mathscr{H}.$$

(iii) $\sum_{G\in\mathscr{H}} |\psi(G)(k)| < \delta$ *for all* $k \in (\bigcup \{G_\gamma: \gamma < \omega^{\alpha\beta}\})^c$.

(iv) $\sum_{G\in\mathscr{H}, G\cap H=\varnothing} |\psi(G)(k)| < \delta$ *for all* $k \in H$, $H \in \mathscr{H}$.

(v) $\sum_{G\in\mathscr{H}} |\psi(G)(k) - \sum_{H\supsetneq G} \psi(H)(k)| < 3 + \delta$ *for all* $k \in \bigcup \{G_\gamma: \gamma < \omega^{\alpha\beta}\} - \bigcup \{F_\gamma: G_\gamma \in \mathscr{H}\}$.

Moreover, if $\mathscr{G}$ *has* δ*-measures* $\{\mu_\gamma: \gamma < \omega^{\alpha\beta}\}$ *then* $\mathscr{H}$ *has* δ*-measures* $\{\mu_\gamma: G_\gamma \in \mathscr{H}\}$.

Proof. We will prove this by induction on β. For convenience we will assume throughout that α is a limit ordinal and that $\alpha_n \uparrow \alpha$. The successor case is proved similarly.

Let (δ_r) be a sequence of positive numbers such that

$$\sum_{r=1}^{\infty} \delta_r < \delta/5.$$

If $\beta = 1$, let $\{G_{\gamma(n)}: n \in \mathbb{N}\}$ be a sequence of disjoint maximal sets in $\mathscr{G}$ such that $G_{\gamma(n)} \in \mathscr{G}_{\alpha_n'}$, $\alpha_n' \geqslant \alpha_n$. Because $\mathscr{W}(\varepsilon, B_{X^\perp}) < \alpha$, for each $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $\mathscr{W}(2^{-1}\delta_r, B_{X^\perp}) < \alpha_n$. Thus by Lemma 3.2 and the fact that $\mathscr{H}_n = \{G_\gamma: G_\gamma \subset G_{\gamma(n)}\}$ contains an α_n-family, there exist a set $G_{\tau(n)} \subset G_{\gamma(n)}$ and a function $h_n \in X$ such that

$$|h_n(k) - f_n(k)| < \delta_r \qquad \text{for} \quad k \in F_{\tau(n)} \cup G^c_{\gamma(n)}$$

and $\|h_n\| < 3$, where f_n is a continuous function on K such that $f_n(k) = f(k)$ for $k \in F_{\gamma(n)}$, $f_n(k) = 0$, for $k \in G^c_{\gamma(n)}$, and $\|f_n\| \leqslant 1$.

Clearly the map ψ: $\{G_{\tau(n)}: n \in \mathbb{N}\} \to X$ given by $\psi(G_{\tau(n)}) = h_n$ defines a $(3 + \delta)$-bounded $\hat{1}$-family of functions in X satisfying (i)–(v).

Next assume that the proposition has been established for all $\beta' < \beta$. If $\beta = \beta' + 1$, for some β' then $\bigcup \{\mathscr{G}_\gamma: \alpha\beta' \leqslant \gamma < \alpha\beta\}$ is an $\hat{\alpha}$-family and thus by the case $\beta = 1$ there is a $(3 + 5^{-1}\delta)$-bounded $\hat{1}$-family of functions in X, $(\psi, \mathscr{H})$, satisfying (i)–(v).

For each $G_{\tau(n)} \in \mathscr{H}$, $\{G_\gamma: G_\gamma \subsetneq G_{\tau(n)}\}$ is an $\widehat{\alpha\beta'}$-family and thus by the induction hypothesis there is a $(7 + \delta_n)$-bounded β'-family of functions in X, $(\psi_n, \mathscr{H}_n)$, satisfying (i)–(v). Clearly $\mathscr{H}' = \mathscr{H} \cup (\bigcup \mathscr{H}_n)$ is a β-family and the map ψ': $\mathscr{H}' \to X$ defined by

$$\begin{aligned} \psi'(H) &= \psi(H) && \text{if} \quad H \in \mathscr{H} \\ &= \psi_n(H) && \text{if} \quad H \in \mathscr{H}_n, \quad n \in \mathbb{N} \end{aligned}$$

defines a $\hat{\beta}$-family of functions in X. The proof that $(\psi', \mathscr{H}')$ is a $(7 + \delta)$-bounded $\hat{\beta}$-family of functions in X satisfying (i)–(v) is similar to the proof of Proposition 2.3, and we leave it to the reader.

If β is a limit ordinal and $\beta_n \uparrow \beta$, then there are disjoint maximal sets $G_{\gamma(n)}$

in $\mathscr{G}$ with $G_{\gamma(n)} \in \mathscr{G}_{\alpha\beta'_n}$, $\beta'_n > \beta_n$, for all $n \in \mathbb{N}$. For each n, $\{G_\gamma: G_\gamma \subset G_{\gamma(n)}\}$ contains an $\widehat{\alpha(\beta_n+1)}$-family, and thus by the inductive hypothesis there is a $(7+\delta_n)$-bounded $\widehat{\beta_n+1}$-family of functions $(\psi_n, \mathscr{H}_n)$ in X satisfying (i)–(v). Clearly we can replace $\mathscr{H}_n$ by a β_n-family $\mathscr{H}'_n \subset \mathscr{H}_n$ for each n and thus $\mathscr{H}' = \bigcup \{\mathscr{H}'_n: n \in \mathbb{N}\}$ is a $\hat{\beta}$-family. By defining $\psi': \mathscr{H}' \to X$ by $\psi'(H) = \psi_n(H)$ for all $H \in \mathscr{H}'_n$, $n \in \mathbb{N}$, we have a $\hat{\beta}$-family of functions in X. It is easy to see that $(\psi', \mathscr{H}')$ is a $(7+\delta)$-bounded $\hat{\beta}$-family of functions in X satisfying (i)–(v).

The "moreover" assertion can be obtained by proving at each stage of the induction that if we have $\{\mu_\gamma: \gamma \leqslant \omega^{\alpha\beta}\}$ then $\mathscr{H}$ can be chosen such that $\{\mu_\gamma: G_\gamma \in \mathscr{H}\} \cup \{\mu_{\omega^{\alpha\beta}}\}$ is homeomorphic to $[1, \omega^\beta]$.

Remark 3.0. By using the more quantitative version of Lemma 3.1 which results from the proof, i.e., $|\langle \mu, \phi f\rangle| < \eta - \tau \Rightarrow \|h\| < 1 + \eta\tau^{-1}$, and the fact the contradiction is obtained from functions of the form $f + s^{-1}\sum_{j=1}^n \gamma_j(f - w_j)$, $w_j \in w_\tau$, $\sum_{j=1}^n |\gamma_j| = 1$, $s^{-1} < \eta\tau^{-1}$, the following holds:

LEMMA 3.1′. *Let X be a subspace of $C(K)$ for some compact Hausdorff space K and let E and F be disjoint closed subsets of K. Let $f \in C(K)$ such that $|f(k)| = 1$ for all $k \in E$, $f(k) = 0$ for all $k \in F$. Suppose that there is an $\eta > 0$ such that for all $\mu \in X^\perp$, $\|\mu\| \leqslant 1$, there is a function $\phi \in C_{\mathbb{R}}(K)$, $1 \geqslant \phi \geqslant 1_E$, such that $|\langle \mu, \phi f\rangle| < \eta^2(1+\eta)^{-1}$. Then there is a function $h \in X$ such that $|h(k) - f(k)| < \eta$ for all $k \in E \cup F$ and $\|h - f\| < 1 + \eta$.*

Proof. Let $\tau = \eta(1+\eta)^{-1}$ in the proof of Lemma 3.1.

By using Lemma 3.1′ in place of Lemma 3.1 it follows from the proofs of Lemma 3.2 and Proposition 3.3 that we can have a $(3+\delta)$-bounded $\hat{\beta}$-family of functions in X rather than a $(7+\delta)$-bounded $\hat{\beta}$-family.

PROPOSITION 3.4. *Let X be a subspace of $C(K)$ and let $J: X \to C(K)$ be the inclusion. Suppose that B is a w^*-closed subset of B_{X^*} and that $\mathscr{W}(\varepsilon, B_{X^\perp}) < \alpha$, for every $\varepsilon > 0$. If there is an $\widehat{\alpha\omega^\beta}$-family of subsets of K with ε-measures in a subset A of $B_{C(K)^*}$ such that for some $\delta < 1$ and for all $a \in A$, $\|J^*a\| \geqslant (1-\delta)\|a\|$ and $J^*A \subset B$, then there is a subspace Y of X such that Y is isomorphic to $C_0(\omega^{\omega^\beta})$ and B norms Y.*

Proof. Let $\rho > 0$ and let $\mathscr{G} = \{G_\gamma: \gamma < \omega^{\alpha\omega^\beta}\}$ be the $\widehat{\alpha\omega^\beta}$-family with ε-measures $\{\mu_\gamma: \gamma < \omega^{\alpha\omega^\beta}\}$ in A. For each $\gamma < \omega^{\alpha\omega^\beta}$ let F_γ be a closed subset of G_γ such that $\bigcup \{G_\tau: G_\tau \subsetneq G_\gamma\} \subset F_\gamma$ and $|\mu_\gamma|(F_\gamma) > |\mu_\gamma|(G_\gamma) - (1-\delta)\rho 8^{-1}$. Let $\beta_n \uparrow \omega^\beta$ and let $\{G_{\gamma(n)}: n \in \mathbb{N}\}$ be a sequence of disjoint maximal sets in $\mathscr{G}$ such that $\mathscr{H}_n = \{G_\gamma: G_{\gamma(n)} \supset G_\gamma\}$ contains an $\widehat{\alpha\beta_n}$-family. By Lemma 2.5 of [1] and Lemma 0.1 for each $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that there is a $\alpha\beta_k$-family $\mathscr{H}'_k \subset \mathscr{H}_n$ with ε-measures $\{\nu_\tau: \tau \leqslant \omega^{\alpha\beta_k}\}$ in $\{\mu_\gamma: G_\gamma \in \mathscr{H}_n\}$ such that

$|\|v_\tau\| - \|v_{\tau'}\|| < (1-\delta)\rho 8^{-1}$ for all $\tau, \tau' \leqslant \omega^{\alpha\beta_k}$. Choose $f_k \in C(K)$ such that $\|f_k\| = 1$, $|f_k(t)| = 1$ for $t \in \bigcup \mathscr{H}'_k$, and $\langle v_{\omega^{\alpha\beta_k}}, f_k\rangle > \|v_{\omega^{\alpha\beta_k}}\| - (1-\delta)\rho 8^{-1}$. We may assume, by replacing $\{v_\tau : \tau \leqslant \omega^{\alpha\beta_k}\}$ with a suitable neighborhood of $v_{\omega^{\alpha\beta_k}}$, that $\langle v_\tau, f_k\rangle > \|v_{\omega^{\alpha\beta_k}}\| - (1-\delta)\rho 8^{-1}$ for all $\tau \leqslant \omega^{\alpha\beta_k}$.

Let $\sum_{k=1}^{\infty} \rho_k < (1-\delta)\rho 8^{-1}$. By Proposition 3.3 there is a $(7+\rho_k)$-bounded $\widehat{\beta}_k$-family $(\psi_k, \mathscr{I}_k)$ satisfying

(i) $\mathscr{I}_k \subset \mathscr{H}'_k$.

(ii) $|\psi_k(G_\gamma)(t) - f_k(t)| < \rho_k$ for all $t \in F_\gamma$ and $|\psi_k(G_\gamma)(t)| < \rho_k$ for all $t \in G_\gamma^c$, $G_\gamma \in \mathscr{I}_k$.

(iii) $\sum_{G \in \mathscr{I}_k} |\psi_k(G)(t)| < \rho_k$ for all $t \in (\bigcup \{G_\gamma : \gamma < \omega^{\alpha\beta}\})^c$.

(iv) $\sum_{G \in \mathscr{I}_k, G \cap H = \varnothing} |\psi_k(G)(t)| < \rho_k$ for all $t \in H$, $H \in \mathscr{I}_k$.

(v) $\sum_{G \in \mathscr{I}_k} |\psi_k(G)(t) - \sum_{H \sqsupset G} \psi_k(H)(t)| < 3 + \rho_k$ for all $t \in \bigcup \mathscr{H}'_k - \bigcup \{F_\gamma : G_\gamma \in \mathscr{I}_k\}$.

Let $\mathscr{I} = \bigcup \{\mathscr{I}_k : k \in N\}$ and define $Z: \mathscr{I} \to X$ by $Z(G) = \psi_k(G)$ for all $G \in \mathscr{I}_k$, $k \in N$. Because $\sum_{k=1}^{\infty} \rho_k < (1-\delta) \in 8^{-1}$ and $(\psi_k, \mathscr{I}_k)$ satisfies (i)–(v) for each k, it follows that $(Z, \mathscr{I})$ is a $(7 + (1-\delta)\rho 8^{-1})$-bounded $\widehat{\omega^\beta}$-family. Define a map $\Lambda: \mathscr{I} \to X^*$ by $\Lambda(G_\gamma) = J^* v_{\tau(\gamma)}$ for all $G_\gamma \in \mathscr{I}_k$, $k \in N$. ($v_{\tau(\gamma)} = \mu_\gamma$ for $G_\gamma \in \mathscr{I}_k$.) If $G_\gamma \in \mathscr{I}_k$, $\langle \Lambda(G_\gamma), Z(G_\gamma)\rangle = \langle v_{\tau(\gamma)}, \psi_k(G_\gamma)\rangle = \langle v_{\tau(\gamma)}, f_k\rangle - \langle v_{\tau(\gamma)}, f_k - \psi_k(G_\gamma)\rangle$. For $t \in F_\gamma$, $|f_k(t) - \psi_k(G_\gamma)(t)| < \rho_k$, by (ii); for $t \in G_\gamma - F_\gamma$, $|f_k(t) - \psi_k(G_\gamma)(t)| \leqslant 4$; and for $t \in G_\gamma^c$, $|f_k(t) - \psi_k(G)(t)| \leqslant 1 + \rho_k$.

Thus

$$\begin{aligned}
&\langle v_{\tau(\gamma)}, \psi_k(G_\gamma)\rangle \\
&\quad \geqslant \|v_\omega \alpha\beta_k\| - (1-\delta)\rho 8^{-1} - \rho_k |v_{\tau(\gamma)}|(F_\gamma) \\
&\qquad - 4|v_{\tau(\gamma)}|(G_\gamma \cap F_\gamma^c) - (1+\rho_k)|v_{\tau(\gamma)}|(G_\gamma^c) \\
&\quad \geqslant \|v_{\tau(\gamma)}\| - (1-\delta)\rho 4^{-1} - \rho_k |v_{\tau(\gamma)}|(F_\gamma) \\
&\qquad - (1-\delta)\rho 2^{-1} - (1+\rho_k)|v_{\tau(\gamma)}|(G_\gamma)^c \\
&\quad \geqslant (1-\rho_k)|v_{\tau(\gamma)}|(F_\gamma) - \rho_k |v_{\tau(\gamma)}|(G_\gamma)^c - 3(1-\delta)\rho 4^{-1} \\
&\quad \geqslant (1-\rho_k)(\varepsilon - (1-\delta)\rho 8^{-1}) - \rho_k |v_{\tau(\gamma)}|(G_\gamma^c) - 3(1-\delta)\rho 4^{-1} \\
&\quad \geqslant \varepsilon - \rho_k(\varepsilon + |v_{\tau(\gamma)}|(G_\gamma^c)) - 7(1-\delta)\rho 8^{-1} \\
&\quad \geqslant \varepsilon - \rho_k(\varepsilon + 1) - 7(1-\delta)\rho 8^{-1} = \varepsilon(\rho).
\end{aligned}$$

Because ρ was arbitrary, and $\varepsilon(\rho) \to \varepsilon$ as $\rho \to 0$, for any $\varepsilon' < \varepsilon$, $\rho' > 0$ there is a $(7+\rho')$-bounded $\widehat{\omega^\beta}$-family $(Z, \mathscr{I})$ of functions in X with ε'-measures in B. Thus by Proposition 1.3 there is a subspace Y of X such that Y is isomorphic to $C_0(\omega^{\omega^\beta})$ and Y is normed by B.

Remark 3.1. Because $J^*B_{C(K)^*} \supset B_{X^*}$, given any subset $\{x_\gamma^*: \gamma \leqslant \omega^{\omega^\tau}\}$ of B_{X^*} satisfying $|\|x_\gamma^*\| - \|x_{\gamma'}^*\|| < \delta$ for all $\gamma, \gamma' \leqslant \omega^{\omega^\tau}$ and some $\delta > 0$, there is a subset $\{\mu_\beta: \beta \leqslant \omega^{\omega^\tau}\}$ of $B_{C(K)^*}$ such that $\|\mu_\beta\| \leqslant \|J^*\mu_\beta\| + \delta$ and $J^*\mu_\beta \in \{x_\gamma^*: \gamma \leqslant \omega^{\omega^\tau}\}$ for all $\beta \leqslant \omega^{\omega^\tau}$. Indeed, for each $\gamma \in [1, \omega^{\omega^\tau}] - [1, \omega^{\omega^\tau}]^{(1)}$ let $\nu_\gamma \in B_{C(K)^*}$ such that $J^*\nu_\gamma = x_\gamma^*$ and $\|\nu_\gamma\| = \|x_\gamma^*\|$. Let $\{\mu_\beta: \beta \leqslant \omega^{\omega^\tau}\}$ be a subset of $\overline{\{\nu_\gamma: \gamma < \omega^{\omega^\tau}\}}^{w^*}$. If $\nu_{\gamma(n)} \to^{w^*} \nu$, and $\gamma(n) \to \gamma$, then

$$\begin{aligned}\varliminf_{n\to\infty} \|x_{\gamma(n)}^*\| = \varliminf_{n\to\infty} \|\nu_{\gamma(n)}\| &\geqslant \|\nu\| \geqslant \|J^*\nu\| \\ &= \|x_\gamma^*\| \geqslant \varliminf \|x_{\gamma(n)}^*\| - \delta.\end{aligned}$$

Hence for all $\beta \leqslant \omega^{\omega^\tau}$; $\|\mu_\beta\| \leqslant \|J^*\mu_\beta\| + \delta$.

The next result is at least formally an improvement of Proposition 3.4. (We say formally because each complemented subspace of $C(K)$ may be isomorphic to some $C(S)$ for S compact metric.)

COROLLARY 3.5. *Let Z be a complemented subspace of $C(K)$ with projection P and let X be a subspace of Z, with inclusion $J: X \to C(K)$, such that $\mathscr{W}(\varepsilon, B_{X^\perp} \cap B_{((I-P)C(K))^\perp}) < \alpha$ for every $\varepsilon > 0$. Let B be a w^* closed subset of B_{X^*}. If there is an $\alpha\omega^\beta$-family of subsets of K with ε-measures in a subset A of $B_{((I-P)C(K))^\perp}$ such that for some $\delta < 1$, for all $a \in A$, $\|J^*a\| \geqslant (1-\delta)\|a\|$ and $J^*A \subset B$, then there is a subspace Y of X such that Y is isomorphic to $C_0(\omega^{\omega^\beta})$ and B norms Y.*

Proof. Let $X_1 = \overline{\mathrm{sp}}(X \cup (I-P)\,C(K))$ and note that $\mathscr{W}(\varepsilon, B_{X_1^\perp}) < \alpha$ for every $\varepsilon > 0$. By Proposition 3.4 there is a subspace Y of X_1 such that Y is isomorphic to $C_0(\omega^{\omega^\beta})$ and B norms Y. Moreover, from the proof, we see that A norms Y as well. Because $A \subset (I-P)\,C(K)^\perp$, it follows that A norms PY, PY is isomorphic to $C_0(\omega^{\omega^\beta})$, $PY \subset PX_1 = X$, and hence that PY is the required subspace of X.

An ordinal γ is said to be prime if $\gamma = \alpha + \beta$, $\beta \neq 0$, implies that $\beta = \gamma$.

THEOREM 3.6. *Let X be a subspace of $C(K)$ and let $J: X \to C(K)$ be the inclusion. Let γ be a prime ordinal such that* $\sup\{\mathscr{W}(\varepsilon, B_{X^\perp}): \varepsilon > 0\} < \omega^\gamma$ *and let B be a w^* closed convex symmetric subset of B_{X^*}. A necessary and sufficient condition for B to norm a subspace Y of X such that Y is isomorphic $C_0(\omega^{\omega^\alpha})$, $\alpha \geqslant \gamma$, is that there exist $\varepsilon > 0$, $\delta < 1$, a subset A of $B_{C(K)^*}$ such that $J^*A \subset B$ and $\|J^*a\| \geqslant (1-\delta)\|a\|$ for all $a \in A$, and an $\widetilde{\omega^\alpha}$-family of open subsets of K with ε-measures in A.*

Proof. Combine Proposition 1.7 and Proposition 3.4.

Remark 3.2. We could obviously formulate a corollary to Theorem 3.6 of a similar nature to Corollary 3.5, but we leave this to the reader.

Our next result ties together our results and those of Rosenthal [22].

PROPOSITION 3.6. *Let X be a subspace of $C(K)$, K compact metric such that $X^\perp$ is separable and let $J: K \to C(K)$ be the inclusion. Suppose that B is a w^* closed convex symmetric subset of B_{X^*}. The following are equivalent:*

(a) *B norms a subspace Y of X such that Y is d-isomorphic to $C[0, 1]$, i.e., there is an isomorphism $T: C[0, 1] \to Y$ with $\|T\| \|T^{-1}\| \leqslant d$.*

(b) *For every $\alpha < \omega_1$, B norms a subspace of X such that Y is d-isomorphic to $C_0(\omega^{\omega^\alpha})$ (d may depend on α).*

(c) *For every $\alpha < \omega_1$ there exist $\varepsilon > 0$, $\delta < 1$, a subset A of $B_{C(K)^*}$ such that $J^*A \subset B$ and $\|J^*a\| \geqslant (1-\delta)\|a\|$ for all $a \in A$, and an $\widehat{\omega^\alpha}$-family of open subsets of K with ε-measures in A.*

(d) *B is non-separable.*

Proof. We will prove the following implications (a) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Leftrightarrow$ (b) $\Rightarrow$ (a). First note that because $X^\perp$ is separable, the Szlenk index of $X^\perp$ is countable. (See [24].) Because the Szlenk index dominates $\mathscr{W}(\varepsilon, B_{X^\perp})$ there is an ordinal $\gamma < \omega_1$ such that $\mathscr{W}(\varepsilon, B_{X^\perp}) < \gamma$, for all $\varepsilon > 0$.

(a) $\Rightarrow$ (d). If T is an isomorphism from $C[0, 1]$ onto Y then $aT^*B \supset B_{C[0,1]^*}$ for some $a < \infty$. Thus any subset $\{b_t : t \in [0, 1]\}$ of B such that $aT^*b_t = \delta_{\{t\}}$ for all $t \in [0, 1]$ will be an uncountable $(a\|T\|)^{-1}$ separated set.

(d) $\Rightarrow$ (c). A result of Rosenthal [22] states that if B_1 is a nonseparable w^*-closed convex symmetric subset of $B_{C(K)^*}$, B_1 norms a subspace of $C(K)$ isometric to $C(\Delta)$, where Δ is the Cantor set. Let $B_1 = J^{*-1}(B) \cap B_{C(K)^*}$. Because $C(\Delta)$ contains subspaces isometric to $C_0(\omega^{\omega^\alpha})$ for every $\alpha < \omega_1$, there is an $\widehat{\omega^\alpha}$-family $\{G_\rho : \rho < \omega^{\omega^\alpha}\}$ of subsets of K with ε-measures $\{\mu_\rho : \rho < \omega^{\omega^\alpha}\}$ in B_1. By Lemma 2.4 of [1] one of the sets $D = \{\rho: \|J^*\mu_\rho\| \geqslant (\varepsilon/2)\|\mu_\rho\|\}$ and $E = \{\rho: \|J^*\mu_\rho\| < (\varepsilon/2)\|\mu_\rho\|\}$ must contain a subset homeomorphic to $[1, \omega^{\omega^\alpha})$. If it were E then for each $\rho \in E$ there would be an element $\nu_\rho \in X^\perp$ such that $\|\nu_\rho - \mu_\rho\| < (\varepsilon/2)\|\mu_\rho\|$. However, this implies that $|\nu_\rho|(G_\rho) \geqslant (\varepsilon/2)$ for all $\rho \in E$ and that if $\nu_{\rho(n)} \to \nu$ and $\rho(n) \to \rho$ then $\underline{\lim}\, \|\nu_{\rho(n)} - \mu_{\rho(n)}\| \geqslant \|\nu - \mu_\rho\|$ and $|\nu|(G_\rho) \geqslant \varepsilon/2$. Thus $\overline{\{\nu_\rho : \rho \in E\}}^{w^*}$ would contain a set of $\varepsilon/2$-measures for an $\widehat{\omega^\alpha}$-family contained in $\{G_\rho : \rho < \omega^{\omega^\alpha}\}$—contradicting $\mathscr{W}(\varepsilon/2, B_{X^\perp}) < \gamma$, if $\omega^\alpha > \gamma$. Hence if $\omega^\alpha > \gamma$, by Lemma 0.1, there is an $\widehat{\omega^\alpha}$-family with ε-measures in $\{\mu_\rho : \rho \in D\}$, establishing (c).

(b) $\Leftrightarrow$ (c). This follows immediately from Proposition 3.4 and Proposition 1.7.

(b) $\Rightarrow$ (a). (This is a minor modification of an argument of Bourgain [4].)

As before let Δ be the Cantor set. Consider the set $\mathscr{S}_{\varepsilon,d} = \{(S, F)$:

$S\colon C(\Delta)\to X$, $\|S\|\leqslant 1$, F is a closed subset of Δ and for all $x\in C(\Delta)$, $\|Sx\|\geqslant d^{-1}\|x_{|F}\|_\infty$ and $\sup\{|\langle b, Sx\rangle|\colon b\in B\}\geqslant \varepsilon\|x_{|F}\|_\infty\}$.

Note that if Y is a subspace of X such that Y is isomorphic to $C(\omega^{\omega^\alpha})$ with isomorphism $T\colon C(\omega^{\omega^\alpha})\to Y$, $\|T\|\leqslant 1$, there is a subset F of Δ homeomorphic to $[1,\omega^{\omega^\alpha}]$ and an isometry $W\colon C(F)\to C(\omega^{\omega^\alpha})$ such that $TWR_F\colon C(\Delta)\to Y$ is an isomorphism onto Y, where R_F is the restriction to F. If $\|T^{-1}\|\leqslant d$ and B norms Y, then $(TWR_F, F)\in\mathscr{S}_{\varepsilon,d}$ for some $\varepsilon>0$. Let $\mathscr{D}$ be the set of all closed subsets of Δ in the Hausdorff topology and let $\mathscr{B}$ be the operators of norm less than or equal to one from $C(\Delta)$ into X with the topology of pointwise convergence. It follows from standard results that $\mathscr{B}\times\mathscr{D}$ is a Polish space (i.e., a complete separable metric space) and it is easily checked that $\mathscr{S}_{\varepsilon,d}$ is a closed subset of $\mathscr{B}\times\mathscr{D}$. Thus the projection of $\mathscr{S}_{\varepsilon,d}$ into $\mathscr{D}$ is an analytic set. For each $\alpha<\omega_1$, by (b), there is a pair $(S_\alpha, F_\alpha)\in\mathscr{S}_{\varepsilon(\alpha),d(\alpha)}$ for some $\varepsilon(\alpha)>0$, $d(\alpha)<\infty$ with $F^{(\omega^\alpha)}\neq\varnothing$. Hence for some $\varepsilon_0>0$, $d_0<\infty$, $\mathscr{S}_{\varepsilon_0,d_0}$ contains pairs for α arbitrarily large. We claim that $\mathscr{S}_{\varepsilon_0,d_0}$ contains a pair (S,F) with $F^{(\omega_1)}\neq\varnothing$. Indeed, a result of Hillard [11] or [6] shows that an analytic subset D of $\mathscr{D}$ which contains only countable subsets of Δ has an upper bound, i.e., there is an ordinal $\beta<\omega_1$ such that for all $F\in D$, $F^{(\beta)}=\varnothing$. Thus the required pair (S,F) with $F^{(\omega_1)}\neq\varnothing$ exist and it follows that $S(C(\Delta))\subset X$ contains a subspace d-isomorphic to $C[0,1]$ and ε normed by B.

4. Remarks and Questions

The results of [1] and Section 3 provide a partial solution to the following:

Problem. If X is a subspace of $C(K)$, K a compact metric space, does at least one of X and $C(K)/X$ have a subspace isomorphic to $C(K)$?

For the cases of c_0 and $C[0,1]$ the answer is yes. The first follows from results of Pelczynski [16] and the fact that at least one of X and c_0/X is infinite dimensional. The second is a theorem of Lindenstrauss and Pelczynski [14]. For X a complemented subspace of $C(K)$ the answer is also yes [2].

Our results imply the following positive result:

Proposition 4.1. *Let X be a subspace of $C(\omega^{\omega^\gamma})$ for some prime ordinal $\gamma<\omega_1$.*

(a) *If there is an $\varepsilon>0$ such that $\mathscr{W}(\varepsilon, B_{X^\perp})\geqslant\omega^\gamma$, $C(\omega^{\omega^\gamma})/X$ has a subspace isomorphic to $C_0(\omega^{\omega^\gamma})$.*

(b) If $\sup\{\mathscr{W}(\varepsilon, B_{X^\perp}): \varepsilon > 0\} < \omega^\gamma$ *then* X *has a subspace isomorphic to* $C_0(\omega^{\omega^\gamma})$.

Proof. Part (a) is immediate from Corollary 0.5 of [1].

For part (b), let $\varepsilon > 0$ and consider the sets $D_\epsilon = \{\alpha: \|\delta_\alpha\|_{X^*} > 1 - \varepsilon\}$ and $E_\epsilon = \{\alpha: \|\delta_\alpha\|_{X^*} \leqslant 1 - \varepsilon\}$. By Lemma 2.4 of [1] at least one of these sets contains a subset F homeomorphic to $[1, \omega^{\omega^\alpha})$. Suppose $F \subset E_\epsilon$. Then if $\varepsilon > \tau > 0$ for each $\alpha \in F$ there is a measure $\mu_\alpha \in X^\perp$, $\|\mu_\alpha\| = 1$, such that $\|\delta_\alpha - \mu_\alpha\| < 1 - \varepsilon + \tau$. Let $\mathscr{G} = \{G_\alpha : \alpha < \omega^{\omega^\gamma}\}$ be an $\widehat{\omega^\gamma}$-family of clopen subsets of $[1, \omega^{\omega^\gamma}]$ such that $\alpha \in G_\alpha$.

We claim that there is an $\widehat{\omega^\gamma}$-family $\mathscr{F}$ contained in $\mathscr{G}$ and a subset H of $\overline{\{\mu_\alpha : \alpha \in F\}}^{w^*}$ such that H is a set of $(\varepsilon - \tau)$-measures for $\mathscr{F}$. Indeed, suppose that $(\alpha(n)) \subset F$, $\alpha(n) \to \alpha$ and $\mu_{\alpha(n)} \to^{w^*} \mu$. Because $\|\delta_{\alpha(n)} - \mu_{\alpha(n)}\| < 1 - \varepsilon + \tau$, $\mu_{\alpha(n)}(\{\alpha(n)\}) > (\varepsilon - \tau + 1)/2$. Hence

$$\begin{aligned}\mu(\{\alpha\}) &\geqslant \overline{\lim}\, [\mu_{\alpha(n)}(\{\alpha(n)\}) - (\|\mu_{\alpha(n)}\| - |\mu_{\alpha(n)}(\{\alpha(n)\})|)] \\ &\geqslant \overline{\lim}\, (2\mu_{\alpha(n)}(\{\alpha(n)\}) - \|\mu_{\alpha(n)}\|) \\ &\geqslant \varepsilon - \tau \qquad \text{and} \qquad |\mu|\, (G_\alpha) \geqslant \varepsilon - \tau.\end{aligned}$$

Thus if $H \subset \overline{\{\mu_\alpha : \alpha \in F\}}^{w^*}$, H is homeomorphic to $[1, \omega^{\omega^\gamma})$, and $H - H^{(1)} \subset \{\mu_\alpha : \alpha \in F\}$, a simple induction argument shows that $\{G_\alpha : \alpha \in \{\rho: \mu_\rho \in H\}\}$ contains an $\widehat{\omega^\gamma}$-family with $(\varepsilon - \tau)$ measures in H. This contradicts the assumption that $\mathscr{W}(\varepsilon - \tau, B_{X^\perp}) < \omega^\gamma$. Therefore $D_\epsilon \supset F$ and by Proposition 3.4, X contains a subspace isomorphic to $C_0(\omega^{\omega^\gamma})$.

Thus for γ prime the possibility remains that there is a subspace X of $C(\omega^{\omega^\gamma})$ such that $\sup\{\mathscr{W}(\varepsilon, B_{X^\perp}): \varepsilon > 0\} = \omega^\gamma$ but $\mathscr{W}(\varepsilon, B_{X^\perp}) < \omega^\gamma$ for every $\varepsilon > 0$ and neither X nor $C(\omega^{\omega^\gamma})/X$ contains a subspace isomorphic to $C(\omega^{\omega^\gamma})$. Note that in this case $C(\omega^{\omega^\gamma})/X$ contains a subspace isomorphic to $C(\omega^{\omega^\rho})$ for all $\rho < \gamma$.

If γ is not prime then it is not difficult to see that the answer to the problem is no. Indeed, suppose $\gamma = \beta + \alpha$, $\alpha, \beta < \gamma$. Let $X = \overline{\mathrm{sp}}\{1_{(\tau^-, \tau]} : \tau \in [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho)} - [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho + 1)}, \rho \leqslant \omega^\alpha\}$, where $\tau^- = \sup([1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho)} - [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho + 1)}) \cap [1, \tau)$. It is easy to see that X is isometric to $C(\omega^{\omega^\alpha})$. (Let $\mathscr{G}_\rho = \{(\tau^-, \tau]: \tau \in [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho)} - [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho + 1)}\}$ then $\mathscr{G} = \bigcup \{\mathscr{G}_\rho : \rho \leqslant \omega^\alpha\}$ is an $\widehat{\omega^\alpha}$-family.) We claim that $C(\omega^{\omega^\gamma})/X$ is isomorphic to $C(\omega^{\omega^\beta})$. For each $\tau \in [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho)} - [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho + 1)}$, $\rho \in [1, \omega^\alpha] - [1, \omega^\alpha]^{(1)}$, let $\tilde{Y}(\tau, \rho) = \{f \in C_0(\tau^-, \tau]: f$ is constant on $(\xi^-, \xi]$ for $\xi \in (\tau^-, \tau]^{(\omega^\beta(\rho - 1) + 1)}\}$. $\tilde{Y}(\tau, \rho)^* \sim \overline{\mathrm{sp}}\{\delta_\tau - \delta_\xi : \xi \in (\tau^-, \tau]^{(\omega^\beta(\rho - 1) + 1)}\} = (C[\tau^-, \tau]/X)^*$. $\tilde{Y}(\tau, \rho)$ is isometric to $C_0(\omega^{\omega^\beta})$ and $\tilde{Y}(\tau, \rho) \sim Y(\tau, \rho) = C(\tau^-, \tau]/X$. $(C(\omega^{\omega^\gamma})/X)^* = \overline{\mathrm{sp}}\{\bigcup Y(\tau, \rho)^* : \tau \in [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho)} - [1, \omega^{\omega^\gamma}]^{(\omega^\beta\rho + 1)}, \rho \in [1, \omega^\alpha] - [1, \omega^\alpha]^{(1)}\}$ and this is w^* isometric to $(\sum_{\tau,\rho} Y(\tau, \rho))^*_{c_0}$. This follows from the fact that the measures in $Y(\tau, \rho)^*$ have supports disjoint from the support of the measures in $Y(\tau',\rho')^*$, for $(\tau,\rho)\neq(\tau',\rho')$. Thus $C(\omega^{\omega^\gamma})/X$ is isometric to $(\sum_{\tau,\rho} Y(\tau,\rho))_{c_0}$ which is isometric to $C(\omega^{\omega^\beta})$.

As a second application of the results of Section 3 we have the following.

PROPOSITION 4.2. *There is a Banach space X which is not primary but has the property that if $Y\subset X$ either X/Y or Y contains a subspace isomorphic to X.*

Proof. Let Z be a subspace of c_0 which is not isomorphic to a complemented subspace of $C[0,1]$, e.g., Z is a subspace of c_0 failing the approximation property. Let $X=C[0,1]\oplus Z$. Clearly X is not primary. However, X is isomorphic to a subspace of $C[0,1]\oplus c_0\sim C[0,1]$ with separable annihilator. If Y is a subspace of X, then either X/Y has a separable dual, in which case Y is isomorphic to a subspace of $C[0,1]$ with a separable annihilator, or X/Y has nonseparable dual. In the first case Proposition 3.6 (d) $\Rightarrow$ (a) with $B=B_{Y^*}$ shows that Y contains a subspace isomorphic to $C[0,1]$ and hence a subspace isomorphic to X. In the second case with $B=B_{Y^\perp}$ Proposition 3.6 implies that there is a subspace W of X normed by B such that W is isomorphic to $C[0,1]$. This implies that $Q(W)$ is a subspace of X/Y isomorphic to $C[0,1]$, where Q is the quotient map of X onto X/Y. Thus X/Y has a subspace isomorphic to X.

Our motivation for Section 3 was the successful characterization of $C(\omega^{\omega^\alpha})$ preserving operators on the disc algebra. However, it should be noted that not all of the results for the disc algebra are consequences of the general results of Section 3.

In particular it is obvious that $\mathscr{W}(\varepsilon,B_{\mathscr{A}^\perp})>0$ and thus the results of Section 3 do not apply to subsets of $B_{C(\partial D)^*}$ which norm a subspace of $\mathscr{A}$ isomorphic to c_0. Thus the result of Delbaen and Kisliakov is not a consequence of these results.

On the other hand as a result of a conversation with J. Arazy we have shown that $\mathscr{W}(\varepsilon,B^\perp_{\mathscr{A}})<\omega$. Indeed, it follows from Propositions 3.2 and 3.3 of [13] that $C(\omega^n)$ does not embed uniformly in $C(\partial D/\mathscr{A})$ and thus the ε-Szlenk index of $B^\perp_{\mathscr{A}}$ is finite for every $\varepsilon>0$. This in turn implies that $\mathscr{W}(\varepsilon,B^\perp_{\mathscr{A}})<\omega$ for every $\varepsilon>0$.

An examination of the proof of Theorem 7.1 of [17] shows that the properties of $\mathscr{A}$ as a uniform algebra play an important role. Thus it seems natural to ask if the results can be improved in the context of uniform algebras where the algebra is considered as a subspace of the continuous functions on its Shilov boundary. (The maximal ideal space or other boundaries could be used but in that case the annihilator can be quite large.)

It would also be interesting to know if $\mathscr{W}(\varepsilon,B_{X^\perp})$ can be computed for X any of the standard examples of uniform algebras.

Next we will examine the consequences of our results for the classification of the complemented subspaces of subspaces of $C(K)$ with small annihilator and in particular for the disc algebra. Among the complemented subspaces of these spaces are the $C(S)$ spaces. The following proposition gives a criterion for determining if a complemented subspace is isomorphic to a $C(S)$ space.

PROPOSITION 4.3. *Let X be a subspace of $C(K)$, K compact metric, such that $\sup \mathscr{W}(\varepsilon, B_{X\perp}) < \omega^{\delta}$, for some prime ordinal δ, and let $J: X \to C(K)$ be the inclusion.*

Suppose that Y is a complemented subspace of X with projection P.

(1)(a) *If $\alpha \geqslant \delta$, a necessary and sufficient for Y to contain a subspace isomorphic to $C_0(\omega^{\omega^\alpha})$ is that $\mathscr{W}(\varepsilon, J^{*-1}P^*B_{X^*} \cap B_{C(K)^*}) \geqslant \omega^{\alpha}$ for some $\varepsilon > 0$.*

(b) *If Y is isomorphic to a complemented subspace of $C_0(\omega^{\omega^\alpha})$ and $\mathscr{W}(\varepsilon, J^{*-1}B_{X^*} \cap B_{C(K)^*}) \geqslant \omega^{\alpha}$ for some $\varepsilon \geqslant 0$, $\alpha \geqslant \delta$, then Y is isomorphic to $C_0(\omega^{\omega^\alpha})$.*

(2) *If Y^* is non-separable then Y contains a subspace isomorphic to $C[0, 1]$. Consequently if Y is, in addition, isomorphic to a complemented subspace of $C[0, 1]$, Y is isomorphic to $C[0, 1]$.*

Proof. Suppose $\mathscr{W}(\varepsilon, J^{*-1}P^*B_{X^*} \cap B_{C(K)^*}) \geqslant \omega^{\alpha}$, for some $\varepsilon > 0$, $\alpha \geqslant \delta$. By Proposition 2.0 of [1] there is an $\widehat{\omega^{\alpha}}$-family of subsets of K, $\{G_\rho : \rho < \omega^{\omega^\alpha}\}$ with ε-measures in $J^{*-1}P^*B_{X^*} \cap B_{C(K)^*}$, $\{\mu_\rho : \rho < \omega^{\omega^\alpha}\}$. Consider the sets $D = \{\rho: \|J^*\mu_\rho\| \geqslant (\varepsilon/2)\|\mu_\rho\|\}$ and $E = \{\rho: \|J^*\mu_\rho\| < (\varepsilon/2)\|\mu_\rho\|\}$. By Lemma 2.4 of [1] one of D and E must contain a subset homeomorphic to $[1, \omega^{\omega^\alpha})$. Just as in the proof of Proposition 3.6 (d) $\Rightarrow$ (c), if it were E this would imply that $\mathscr{W}((\varepsilon/2), B_{X\perp}) \geqslant \omega^{\alpha}$—a contradiction. Hence D contains a subset D_0 homeomorphic to $[1, \omega^{\omega^\alpha})$, and thus there is an $\widehat{\omega^{\alpha}}$-family of subsets of K with ε-measures in $\{\mu_\rho : \rho \in D_0\}$. By Proposition 3.4, $P^*B_{X^*}$ norms a subspace of X isomorphic to $C_0(\omega^{\omega^\alpha})$ and thus $P(C_0(\omega^{\omega^\alpha}))$ is a subspace of Y isomorphic to $C_0(\omega^{\omega^\alpha})$.

To obtain (b) note that by the results of [18], Y contains a complemented subspace isomorphic to $C_0(\omega^{\omega^\alpha})$. Applying the decomposition method [16], we get that Y is isomorphic to $C_0(\omega^{\omega^\alpha})$.

Part 2 is essentially known (see [17, p 52]), but we will include a proof here. By Proposition 3.6 (d) $\Rightarrow$ (a), $P^*B_{X^*}$ norms a subspace Z of X such that Z is isomorphic to $C[0, 1]$, clearly PZ is the required subspace of Y. As in part (1)(b) it follows that Y contains a complemented subspace isomorphic to $C[0, 1]$. Thus if Y is isomorphic to a complemented subspace of $C[0, 1]$, Y is isomorphic to $C[0, 1]$.

Remark 4.1. Note that the conditions on δ and $\|J_\alpha^*\|$ imposed in Proposition 3.4 do not play a role here. The combinatorial of Lemma 2.4 of [1] can always be used to reduce the problem to considering $J^{*-1}B \cap B_{C(K)}$.

For the special case of the disc algebra Proposition 4.3 identifies the complemented subspaces isomorphic to $C[0, 1]$ or $C(\omega^{\omega^\alpha})$ for $\alpha \geqslant \omega$. Actually in the case of the disc algebra, Corollary 2.5 rather than Proposition 3.4 can be used in the proof of Proposition 4.3 to reduce characterizations of complemented subspaces of A isomorphic to $C_0(\omega^{\omega^\alpha})$ for all $\alpha < \omega_1$.

PROPOSITION 4.4. *Let Y be a complemented subspace of the disc algebra $\mathscr{A}$ with projection P and let $J: \mathscr{A} \to C(\partial D)$ be the inclusion.*

(1)(a) *If $\alpha \geqslant \omega$, a necessary and sufficient condition for Y to contain a subspace isomorphic to $C_0(\omega^{\omega^\alpha})$ is that $\mathscr{W}(\varepsilon, J^{*-1}P^*B_{\mathscr{A}^*} \cap B_{C(\partial D)^*}) \geqslant \omega^\alpha$ for some $\varepsilon > 0$.*

(b) *If $\alpha \geqslant \omega$, then Y is isomorphic to $C_0(\omega^{\omega^\alpha})$ if (and only if) Y is isomorphic to a complemented subspace of $C_0(\omega^{\omega^\alpha})$ and $\mathscr{W}(\varepsilon, J^{*-1}P^*B_{\mathscr{A}^*} \cap B_{C(\partial D)^*}) \geqslant \omega^\alpha$ for some $\varepsilon > 0$.*

(2)(a) *If $\alpha < \omega$, a necessary and sufficient condition for Y to contain a subpsace isomorphic to $C_0(\omega^{\omega^\alpha})$ is that there are constants $\varepsilon > \delta > 0$, a function $j: P^*B_{\mathscr{A}^*} \to J^{*-1}P^*B_{\mathscr{A}^*}$ such that*

$$\|J^*j(b)\| \geqslant (1-\delta)\,\|j(b)\| \qquad \textit{for all} \quad b \in P^*B_{\mathscr{A}^*}$$

and $\mathscr{W}(\varepsilon, j(P^*B_{\mathscr{A}^*}) \geqslant \omega^\alpha$.

(b) *If the condition of (2)(a) is satisfied and Y is isomorphic to a complemented subspace of $C_0(\omega^{\omega^\alpha})$, Y is isomorphic to $C_0(\omega^{\omega^\alpha})$ (and conversely).*

(3) *Y is isomorphic to $C[0, 1]$ if and only if Y^* is non-separable and Y is isomorphic to a complemented subspace of $C[0, 1]$.*

It would be interesting to have some computable invariants to replace the condition that Y be isomorphic to a complemented subspace of $C_0(\omega^{\omega^\alpha})$. Even if Y is isomorphic to a complemented subspace of $C[0, 1]$, no such invariants are known. In particular it is unknown whether or not a complemented subspace Y of $C[0, 1]$ with $\sup \mathscr{W}(\varepsilon, B_{Y^*}) \leqslant \omega^{\alpha+1}$ is isomorphic to a complemented subspace of $C_0(\omega^{\omega^\alpha})$.

In order to classify the all of the complemented subspaces of the disc algebra some new invariants will be needed to classify the spaces with a given index. Unlike the $C[0, 1]$ case, the disc algebra has a complemented subspace of finite index which is not isomorphic to c_0. Indeed, if (x_i) is a basis for $\mathscr{A}$, then $(\sum_{n=1}^\infty [x_i]_{i=1}^n)_{c_0}$ is such a space (see [17, 26]). One problem is to determine whether there is more than one isomorphic type generated in this way.

Note added in proof. It follows from the decomposition method of Pelczynski [16] that the isomorphic type of $(\sum_{n=1}^\infty [x_i]_{i=1}^n)_{c_0}$ is independent of the choice of basis of $\mathscr{A}$.

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